LONG-TERM FLUCTUATION OF ANNUAL SUMS OF PRECIPITATION ON THE TERRITORY OF BULGARIA

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SUMMARY

On the basis of data of 35 rain gauge stations, using the method of polygons, areal annual precipitation sums were calculated for the period of 1896 - 1985 for following geographical units of Bulgaria: a) the Danube hilly plain (region I); b) the foothills of the Stara Planina Mts. (region II); c) high basins of western Bulgaria (region III); d) the Upper Thracian lowland (region IV). The most variable in precipitation is region IV, the least variable region III. Region I and II exhibit the highest degree of similarity of the precipitation régime. In the long-term trend the precipitation series, particularly those for regions II and IV, exhibit a rising trend, the extremes being in accordance with the main maxima and minima of precipitation in a broader European region. Autocorrelation analysis showed a weak persistence of series and insignificant autocorrelation coefficients. From the spectral analysis there followed significant cycles in the length of 15 — 17a (a = year) and cycles in the interval of 2 - 5a. Dynamical MESA and numerical band-pass filtering exhibit a considerable variability of cycles both in phase and in amplitude. The above cycles are also exhibited by coherence analysis of the studied series. From the procession there follow circulation conditioned differences in the precipitation régime between the northern and the southeastern parts of Bulgaria.

1. INTRODUCTION

The objects of the greatest interest in studying the fluctuation of a climate are usually air temperatures and atmospheric precipitation, not only due to long observation series of those elements, but above all for their effect on the environment of the given region and thus on different aspects of human activity. In recent years serious discussions have been held on the intensification of the greenhouse effect due to increasing concentrations of CO_2 and further trace gases in the atmosphere and thus the increase in global air temperature on the Earth. The assumed

warming should, understandably, be reflected also in the changes of the other meteorological elements, above all of atmospheric precipitation (see [11]). The forecast of its changes is, however, in the present general circulation models, biased by a greater degree of uncertainty than the temperature forecast, apart from its great spatial changeability. Therefore a study dealing with the analysis of the statistical structure of the precipitation field in different parts of the Earth does not lose its topical character.

The diagnostics of the precipitation field has also been paid considerable attention to also in the case of Bulgaria. Above all in the 1970s and 1980s there appear numerous papers dealing with the study of longterm precipitation changes, their linkage with the fluctuation of circulation processes on the Northern Hemisphere, the cyclicity of precipitation and the fluctuation of precipitation as the cause of fluctuation of the river runoff (e. g. [8, 13, 17, 21 - 28]). The above papers are based on the analysis of precipitation at the individual stations either from the whole territory of Bulgaria or from smaller regions. Such observations can, however, be biased by local effects, besides systematic and random errors. Therefore it appears appropriate to average the measured values for larger territorial units, although in such series it is not quite possible to eliminate systematic errors (the aerodynamic effect of the rain gauge, wetting, evaporation, etc. — see [20]) that ought to be eliminated already during the primary processing of data by the corresponding meteorological service. In the meantime only corrections of long-term precipitation sums are carried out [9]. Despite this drawback, however, the averaged values can be used for obtaining more general information about the fluctuation of the precipitation.

2. THE CALCULATION OF THE AREAL ANNUAL PRECIPITATION SUMS

The first meteorological stations in Bulgaria were established much later than in most western and central European countries. Thus, the oldest weather station in Sofia started its observations as late as in 1887, whereas the network of weather stations was being formed only in the years 1892 — 1896. Therefore for the calculation of series of areal annual precipitation sums from the territory of Bulgaria data of 35 stations were used for only a ninety-year period of 1896 — 1985. Their geographical position is included in Fig. 1. From the point of view of the physicogeographical regionalization of Bulgaria these stations were divided into four regions:

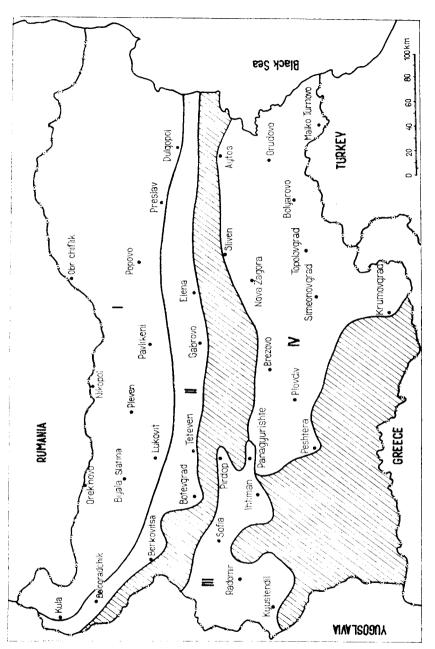


Fig. I. Division of stations on the territory of Bulgaria into regions for which the averaging was performed (the hatched areas are the regions for which the procession was not performed)

- a) Region I (the Danube hilly plain), including 11 stations within elevation of 41 295 m, situated in the northern lowland of Bulgaria. Secular stations are missing in the northeastern part of that region.
- b) Region II (the foothills of the Stara Planina Mts.), constituting a narrow belt of the territory in the foothills of the Stara Planina Mts. with 6 weather stations at the height above sea level of 331 545 m.
- c) Region III (high basins of western Bulgaria), including 6 stations at the elevation of 521 691 m in high mountain basins between the Stara Planina Mts. and the mountain ranges in the southwestern part of Bulgaria (the Rila Mts., the Rodopi Mts., the Pirin Mts.) where stations with long observation series are practically missing.
- d) Region IV (the Upper Thracian lowland), including 12 stations from the region of the Upper Thracian lowland and the hilly parts of the Rodopi Mts., the Sakar Mts. and the Strandzha Mts. with elevation of 56 436 m.

For each region the areal annual precipitation sums for the period of 1896 — 1985 were calculated by the polygon method according to the relation:

$$ar{x}_{\scriptscriptstyle S} = rac{\sum\limits_{i=1}^{n} x\left(arphi_{i}, \; \lambda_{i}
ight) \; s_{i}}{\sum\limits_{i=1}^{n} s_{i}},$$

where s_i is the relative share of the area of the polygon found by planimetry and represented by the precipitation sum of the *i*-th station x (φ_i , λ_i) and n is the number of polygons (Table 1).

The obtained series were further processed by the methods of auto-correlation, spectral and coherence analysis and numerical band-pass filtering in the form described in papers [2, 16].

Table 1. Areal annual precipitation sums (mm) for selected regions of Bulgaria in the period of 1896 — 1985

Year	0	1	2	3	4	5	6	7	8	9
1890							424	769	577	526
1900	753	744	455	539	495	550	797	406	521	611
1910	602	566	652	562	730	656	549	479	440	768
1920	568	475	593	567	754	614	644	634	522	59 0
1930	472	700	471	703	453	552	572	769	520	656
1940	720	732	608	605	763	358	416	537	448	512
1950	423	523	626	488	760	776	664	776	401	563
1960	686	566	559	568	635	498	776	498	540	707
1970	671	674	699	648	491	671	582	568	548	706
1980	652	592	523	490	590	422				

Continuation — Table 1:

Region II

			H	Region	ı II					
Year	0	1	2	3	4	5	6	7	8	9
1890							642	1047	790	673
1900	967	994	555	661	750	714	972	565	669	802
1910	813	766	774	767	947	909	810	605	531	945
1920	752	573	710	687	966	798	744	748	605	793
1930	721	838	572	852	588	730	855	1072	639	849
1940	1017	882	800	804	1066	499	520	709	611	761
1950	587	884	712	698	946	1048	850	1045	567	842
1960	865	699	694	768	931	676	1042	774	793	842
1970	872	808	896	929	731	1061	901	821	783	929
1980	927	918	779	768	671	615				
			F	Region	ı III					
Year	0	1	2	3	4	5	6	7	8	9
1890							608	818	681	573
1900	700	678	400	479	519	511	660	455	492	636
1910	658	562	623	497	701	740	607	497	368	729
1920	544	480	589	579	647	577	611	614	423	613
1920	661	668	478	655	405	675	760	909	519	641
1940	792	652	534	529	742	698	475	593	528	598
1950	472	652	546	537	680	676	664	715	470	698
1960	668	459	553	626	592	496	678	544	592	664
1970	605	610	638	655	560	653	820	554	607	600
1980	649	576	643	546	466	511				·
			j	Regio	n IV					
Year	0	1	2	3	4	5	6	7	8	9
1890							568	818	496	514
1900	862	655	422	450	512	490	739	452	424	509
1910	684	571	519	545	668	676	633	500	569	830
1920	472	422	555	550	552	619	511	511	541	587
1930	562	697	430	622	444	634	638	800	538	632
1930	1012	751	551	662	760	415	550	587	474	500
1940	449	618	656	603	728	880	739	671	525	586
1960	733	622	632	769	555	552	878	520	534	669
	594	757	594	650	532	743	543	604	618	724
1970										

3. STATISTICAL CHARACTERISTICS OF SERIES OF AREAL ANNUAL PRECIPITATION SUMS

The homogeneity of the calculated series of areal annual precipitation sums for four Bulgarian regions was found out by means of the von Neumann test, the Abbe and the Helmert criterions, the Kruskal-Wallis

Table 2. Fundamental statistical characteristics of areal annual precipitation sums for selected regions of Bulgaria in the \tilde{x}_{j0} (\tilde{x}_{j0}) — the lower (upper) decil, \tilde{x}_{23} (\tilde{x}_{73}) — the lower (upper) quartil, \tilde{x}_{50} — median, $d_{\tilde{x}}$ — mean deviation from the mean, s — standard deviation — all data in mm; v — coefficient of variation (v_{0}), c_{α} — coefficient of skewness, c_{ε} — coeffiperiod of 1896 — 1985. Explanation: X — mean precipitation sum, Xmax (Xmin) — the highest (lowest) precipitation sum, ficient of kurtosis, f (0.5),... — frequency of precipitation sums in intervals (\bar{x} —0.5s, \bar{x} +0.5s),...

c_{ε} $f(0.5)$ $f(1.0)$ $f(1.5)$ $f(2.0)$	63.3 82.2 65.6 82.2 70.0 90.0
c_{α}	0.1
s a	18.3 18.1 16.6
s	108 143 100
$d_{\bar{x}}$	90 115 79
3	755 974 702
\widetilde{x}_{75}	671 893 663 668
\widetilde{x}_{50}	575 787 607 590
\widetilde{x}_{25}	520 695 530 521
\widetilde{x}_{10}	453 588 475 452
Year	1945 1945 1918 1945
x_{min}	358 499 368 415
Year	1906 1937 1937 1940
x_{max}	797 1072 909 1012
×	592 790 601 605
Area	Region I Region II Region III Region IV

and the Bartlett tests. With the exception of the Helmert criterion for the series of region IV the homogeneity was confirmed by all methods for all regions.

In Table 2 there are fundamental statistical characteristics of the calculated series. The most variable is annual precipitation in the southeastern part of Bulgaria (region IV), the least variable in the high mountain basins of western Bulgaria (region III). Practically the same variability is exhibited by the remaining two regions in which the course of precipitation coincides best (correlation coefficient $r_{xy} = 0.89$). The fewest common features of precipitation fluctuations exist between regions III and IV ($r_{xy} = 0.64$), as well as between regions I and III ($r_{xy} = 0.67$). The same is the correlation coefficient of region IV with regions I and II (always 0.76), somewhat higher that between regions II and III (0.78).

4. FLUCTUATION AND CYCLICITY OF SERIES OF AREAL ANNUAL PRECIPITATION SUMS — DISCUSSION OF RESULTS

From the graph of three-year running averages of precipitation in Fig. 2 the unsteadiness of precipitation is well observable, asserting itself in both amplitude and time occurrence of the individual partial maxima and minima. Regions II and IV exhibit an overall rising trend of precipitation as can be seen from the graph of eleven-year running averages. All series exhibit maxima in the 1930s which was succeeded by a more conspicuous drop in precipitation in the 1940s. According to [15] the wet years 1930 - 1941 and the dry period of 1942 - 1954 are typical of both the Mediterranean [14] and the western and central Europe south of 50° northern latitude (cf. e. g. [2, 4]). The above increase in precipitation in the 1930s corresponds to above-average frequencies of the occurrence of zonal macrocirculation type E (eastern) on the Northern Hemisphere according to the typification of [10], whereas the drier phase in the subsequent period coincides well with above-average frequencies of the meridional type C. In connection with the rising trend of precipitation in Bulgaria there is a decreasing trend of frequencies of occurrence of the zonal macrotype W (western) and a rising trend of frequencies of the macrotype E (see [5]). According to [12] there is a significant positive correlation between the frequencies of macrotype E and the precipitation of the cold-half year (November — April) in southern Europe, whereas for the frequencies of the W type it is a significant negative correlation. According to [14] the prevalence of meridional circulation over western Mediterranean results in the increase in precipitation, whereas the prevalence of zonal circulation to its reduction.

Up to the 1930s precipitation in Bulgaria was below average, only in the northern part of Bulgaria (region I) it fluctuated around the ave-

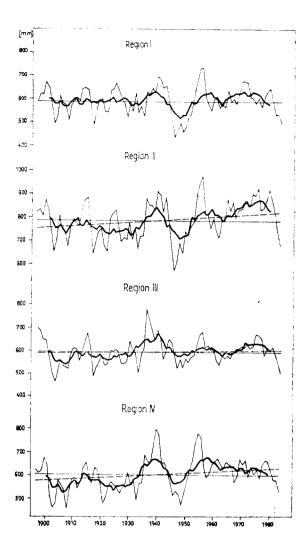


Fig. 2. Long-term fluctuation of series of areal annual precipitation sums for selected regions of Bulgaria in the period of 1896 — 1985 expressed in the form of three-year and eleven-year running averages (horizontally — long-term average, dashed — straight-line trend)

rage. On the other hand, since the 1950s the areal sums are prevailingly above average, with a markedly rising trend in region II and a dropping trend in region IV. That points to a difference in circulation processes affecting the régime of precipitation in the northern and the southeastern parts of Bulgaria.

A schematic representation of circulation factors of the climate is ex-

pressed in Fig. 3. North of the Stara Planina Mts. (but also in basins situated south of them) the highest precipitation sums fall to the summer and the lowest to the winter. The source of precipitation is above all the advection of maritime polar air, when the precipitation falls particularly at the cold fronts with a conspicuous share of precipitation from convection. South of the Stara Planina Mts. the seasonal distribution of precipitation is opposite. The winter maximum there is connected with the activation of cyclonal activity south of Bulgaria, the cyclones affecting more frequently the southern part than the northern one which, on the other hand, is more often under the influence of anticyclonal weather. The precipitation régime is thus the result of operation of dynamic and orographic conditions. Under the influence of orography local cyclones of small horizontal dimensions (microcyclones) can be formed on cold fronts with which heavy precipitation is connected (particularly in the western part of the Upper Thracian lowland and in the northwest of Bulgaria) [6].

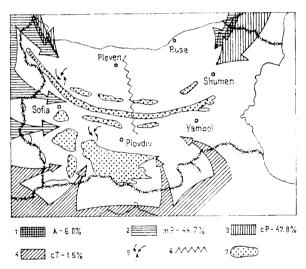


Fig. 3. Schematic representation of circulation factors of the climate of Bulgaria (adapted according to [6]): 1 — arctic air, 2 — maritime polar air, 3 — continental polar air, 4 — continental tropical air, 5 — microcyclones, 6 — inner convergence line, 7 — mountains

Considerable unstability, difference and low persistence of the series studied are pointed to by the results of the autocorrelation analysis (Fig. 4), when autocorrelation coefficients fluctuate prevailingly within 0.2 and -0.2, not differing significantly from zero. The $95^0/_0$ confidence limits are exceeded in only the following cases: region II for the time lag $\tau=15a$ (a = year), region III for $\tau=4a$ and $\tau=28a$, region IV for $\tau=3-4a$ and $\tau=25a$. That would indicate a random character of fluctuation (in

that case r(0) = 1 and the other autocorrelation coefficients are zero). The best expressed persistence is in region IV (r(1) > 0), it is weak for regions II and III (r(1) near zero) and it is missing in region I (r(1) < 0). r(1) not being significant in any of the cases.

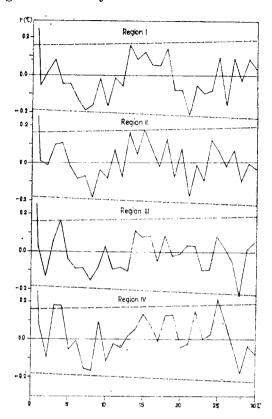


Fig. 4. Values of autocorrelation coefficients $r(\tau)$ of series of areal annual precipitation sums for different regions of Bulgaria (τ — time lag, dashed — 95%0 confidence limits)

The traditional spectral analysis according to Blackman and Tukey gives for northern Bulgaria (regions I, II) a significant cycle in the length of about 14.5a, for region III in the length of about 3.6a and for region IV 3.1a. The maximum entropy spectral analysis (MESA) according to the scheme described in [16], characterized against the preceding method by a better resolution ability, pointed to the existence of an important long-term cycle in all four series, with the length of about 17a (regions II — IV) and 15a (region I) in Fig. 5. Further established significant cycles were as follows: region I 2.9a, 2.2a; region III 4.9a, 3.5a; region IV 4.4a, 3.1a. The cycles found by spectral analysis can be compared to cycles

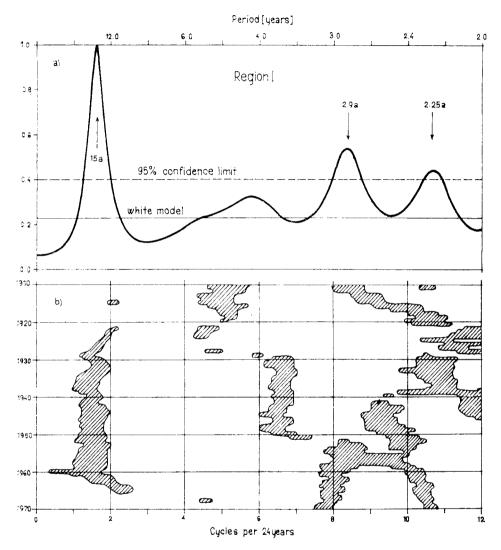
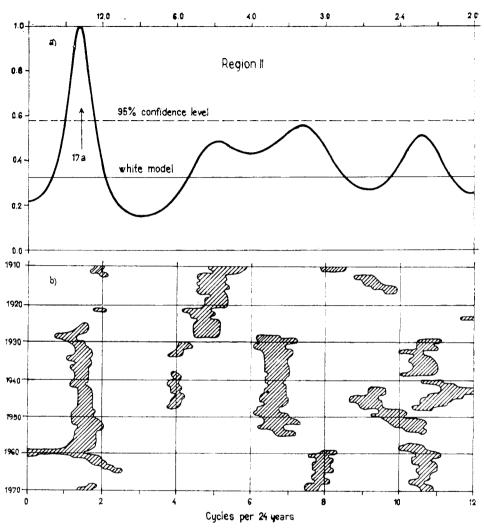
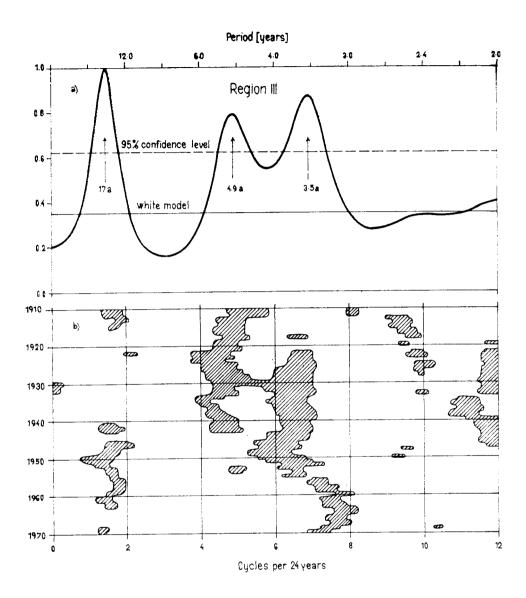
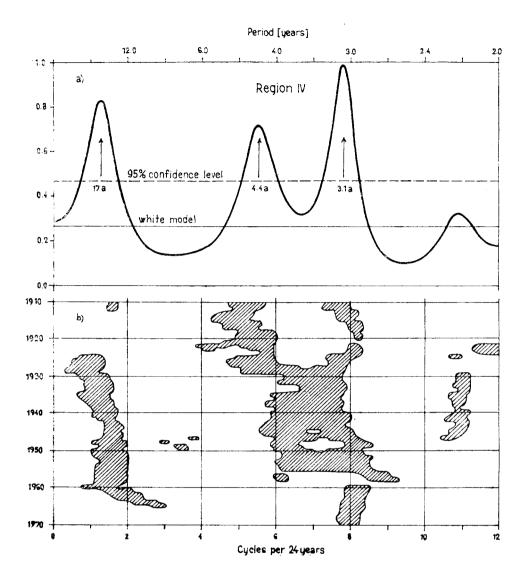


Fig. 5. Maximum entropy spectra (a) and spectrograms (b) of series of areal annual precipitation sums for different regions of Bulgaria









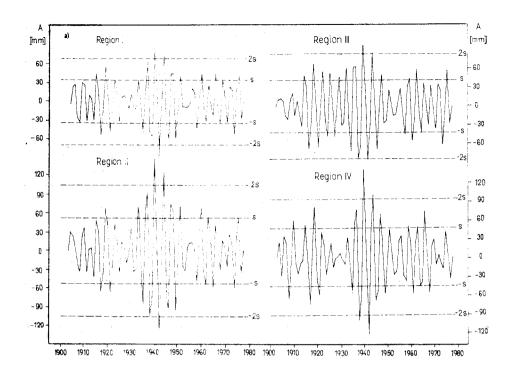
determined for Bulgaria by the method of autocorrelation analysis and integral curves for 37 stations in the period of 1896 — 1975 [26]. For the north of Bulgaria [26] gives the cycle of the length of 13 — 14a, at the lower reaches of the Iskar River 5 — 6a. The performed MESA of the areal sums confirmed the existence of a long-term oscillation (15 — 17a), further significant cycles in region I, however, lying between 2 — 3a. For the southern part of Bulgaria, unlike the analysis carried out in [26] it does not at all give long-term cycles. In the region of high mountain basins of western Bulgaria the cycle of the length of 4a found out by him falls among significant cycles in the length of 4.9 and 3.5a determined by MESA. The significant cycles of the length of 4.4 and 3.1a according to MESA correspond to the area of region IV in the length of 4 — 6a.

The results of spectral analysis correlate to different extents with information of analogical studies from this region. Thus, [14] gives a long-term oscillation in the length of about 20a for western Mediterranean. [18] found significant cycles for Athens in the length of 2-2.3, 2.6-2.8 and 3.5-3.7a, the same as the corresponding peaks of about 2 and 3 years for four further Greek stations. These cycles are in connection with quasi-biennial oscillation and the Southern Oscillation. Similar conclusions were also arrived at in [7] processing 8 Greek rain gauge stations. Significant oscillations in the interval of 2-3a are exhibited, besides further cycles of different length, also by the Rumanian stations Bucharest and Sibiu [3].

From the graphical representation of the established significant cycles in the course of the period studied on the basis of the method of dynamical MESA in the form of the so-called spectrogram (see [16]) there follows a considerable variation and unstability of the cycles found. The spectrograms of regions I and II have most features in common. With the exception of region III a relatively stable occurrence of a long-term cycle is worth mentioning, in shorter cycles a more conspicuous phase fluctuation of significant cycles being typical.

The picture of the temporal variability of the individual cycles can be completed by the results of the numerical band-pass filtering according the method described in [19] (Fig. 6). It appears that the long-term oscillation (for T=15a, i. e. the width of the filter permeability is d=11.25-20a) is more conspicuous in the north of Bulgaria than in the two remaining regions processed. On the other hand, in short-term cycles (for T=4a, d=3-5.3a) a more frequent phase impairment is evident, the oscillation being best expressed in the 1930s and 1940s.

The coincidence of variance spectra of the series studied can be found by means of the coherence analysis. The results of the analysis are evident from Fig. 7. Whereas for regions I and II the coherence values exceed the $99^{\circ}/_{0}$ confidence limit both for long-term (10 — 20a) and for short-term (2 — 6a) oscillations, in the case of further regions they are limited to only certain frequencies. If the significant coherences are summarized



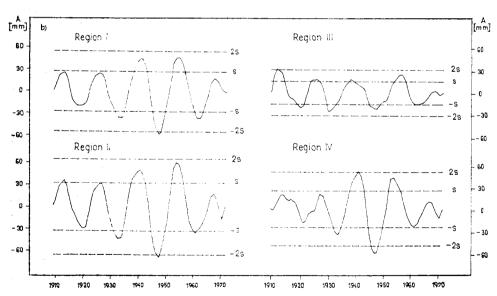


Fig. 6. Numerical band-pass filtering of series of areal annual precipitation sums for different regions of Bulgaria for cycles T=4a (a) and T=15a (b). Explanation: s — standard deviation, A — amplitude

into a frequency graph (Fig. 8), it is evident from it that the most frequented are coherences for cycles of the length of 15-20a (corresponding to the cycle of the length of 15 and 17a, respectively, found in MESA), 5.4a and in the domain of 3.2-4.6a. The phase changes in the spectra between the individual regions are insignificant.

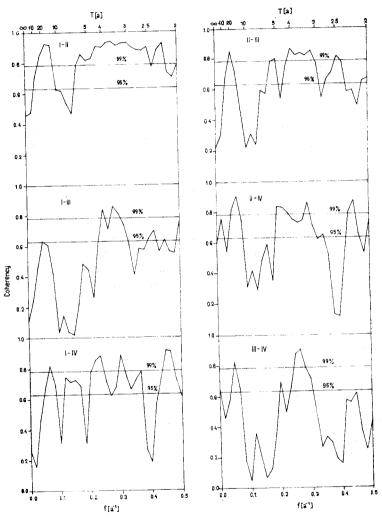


Fig. 7. Coherence analysis of series of areal annual precipitation sums for selected regions of Bulgaria (in the graph indicated 95 and 99% confidence limits; f — frequency, T — length of cycles)

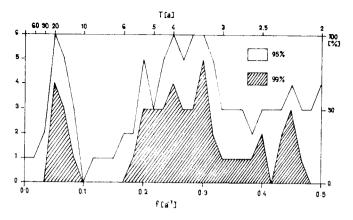


Fig. 8. Frequencies of statistically significant coherence values for series of areal annual precipitation sums for selected regions of Bulgaria

5. CONCLUSION

The analysis of fluctuation of atmospheric precipitation according to the series of areal annual precipitation sums on the territory of Bulgaria in the period of 1896 — 1985 shows quite conspicuous differences particularly between the northern and the southeastern parts of Bulgaria, conditioned by the difference in circulation processes in those regions. Although for the mountain areas of the Stara Planina Mts. and of southwest Bulgaria it was impossible — due to the absence of long-term observation series — to calculate the areal averages, it can be assumed that by the character of the fluctuation of precipitation they will approach the lower situated parts of northern Bulgaria (the Stara Planina Mts.) and/or southern Bulgaria (mountain ranges in the southwestern part of the country). An important fact is the generally increasing precipitation trend (mainly regions II and IV) that would confirm the relation of global temperature rise - precipitation increase. This is in accordance with papers [7] for the territory of Greece and [1] for Northern Hemisphere land according to which the global rise in precipitation operates in the zone 35 — 70° northern latitude.

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 $\{\alpha_{i,j}^{(k)}\}_{i=1}^{k}$