Boundary value problems for nonlinear elliptic equations with a Hardy potential

HABILITATION THESIS

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Abstract

The habilitation thesis is devoted to recent developments on boundary value problems for nonlinear elliptic equations with a Hardy potential in a measure framework. The presence of the Hardy potential which is singular on the boundary of the domain under consideration and the involvement of measures in the analysis yield substantial difficulties and lead to disclose the novelty of the research. New aspects are displayed not only on employed methods but also on observed novel phenomena.

The thesis consists of five chapters. The first chapter addresses the main topics covered in the thesis and presents our contributions, which are collected from our recent works, including results on the existence, nonexistence, uniqueness, a priori estimates and qualitative properties of solutions, a full characterization of isolated boundary singularities, removable singularities. The major features of the problems under investigation depend essentially on the expression of the nonlinear term in equations. Therefore, typical models are successively considered throughout the last four chapters in order to reveal different phenomena. In particular, chapter 2 deals with absorption nonlinear terms and chapter 3 treats source nonlinear terms. Chapter 4 is devoted to an extension of results in the previous chapters to more general equations and systems. Finally chapter 5 focuses on the case where nonlinear terms depend on both solutions and their gradient.

CHAPTER 1

Introduction

Section 1.1 of this chapter is devoted to an overview on boundary value problems for linear and nonlinear elliptic equations involving the classical Laplace operator in function settings and in measure frameworks. We also discuss the role of measures and point out essential differences between the linear case and nonlinear case in measure frameworks. In Section 1.2, we give the motivation for the study of singular operators which are the Laplace operator perturbed by Hardy potentials. Then we address the main problems involving these operators in this thesis. The interaction between Hardy potentials and measures leads to interesting features of the problems and reveals new phenomena. We present briefly our main contributions, including results extracted from our joint paper with Moshe Marcus [**106**], with Konstantinos Gkikas [**78**, **80**] and on our single-author work [**119**], as well as accompanying comments and comparisons with previous in the literature. This may help the reader to grasp the main results more easily and to follow the subsequent sections more smoothly. The detailed statements of these results are provided in Section 1.3 of the chapter for the convenience of the reader. Finally, in Section 1.4, we discuss related and open problems which have recently attracted a great deal of attention.

The thesis is not a self-contained text despite of our effort to make it accessible to researchers and students with different backgrounds. We assume that the reader is familiar with basic notions in functional analysis and measure theory which can be found in standard textbooks, for instance [**34**, **64**, **63**, **1**, **76**]. However, at some places, relevant concepts and ideas from these fields are recalled and explained in order the make the exposition of the main results clearer.

1.1. Overview on boundary value problems

In this section, we first list basic notations that are used frequently throughout the thesis. The reader is referred to the standard textbooks [**1**, **34**, **63**, **76**, **128**, **116**, **139**, **140**] for more properties of these notations. Then we recall well known results for boundary value problems for linear and nonlinear equations involving the classical Laplace operator. These results, as well as the proofs, can be found in excellent references [**116**, **138**, **139**].

Basic notations.

• Assume Ω is a domain (namely a connected, open nonempty subset) in $\mathbb{R}^N$ $(N \geq 1)$. Let $C(\Omega)$ be the space of continuous functions on Ω. We denote by $C^k(\Omega)$ the space of functions k times continuously differentiable on Ω (for integer $k \geq 1$) and $C^\infty(\Omega) = \cap_{k \geq 1} C^k(\Omega)$. Let $C_c(\Omega)$ be the space of continuous functions on Ω with compact support in Ω. Put $C_c^k(\Omega) = C^k(\Omega) \cap C_c(\Omega)$ and $C_c^\infty(\Omega) = C^\infty(\Omega) \cap C_c(\Omega)$.

• Let $\phi \in C(\Omega)$ be a positive weight function. Denote by $L^\kappa(\Omega,\phi)$, $1 \leq \kappa < \infty$) the weighted Lebesgue space of functions v satisfying $\int_\Omega |v|^\kappa \phi dx < \infty$. This space is endowed with the norm

$$\|v\|_{L^\kappa(\Omega,\phi)} := \left(\int_\Omega |v|^\kappa \phi dx \right)^{\frac{1}{\kappa}}.$$

When $\phi \equiv 1$, these spaces become the usual Lebesgue spaces $L^\kappa(\Omega)$. We denote by $L^\kappa_{loc}(\Omega)$ the space of functions v such that $v \in L^\kappa(\Omega')$ for any compact subset $\Omega' \subset \Omega$.

• For $1 \leq \kappa < \infty$, the weighted Sobolev space $W^{m,\kappa}(\Omega,\phi)$ is defined by

$$W^{m,\kappa}(\Omega,\phi) := \{v \in L^\kappa(\Omega,\phi) :\ D^\beta v \in L^\kappa(\Omega,\phi) \text{ for every } |\beta| \leq m\}.$$

This space is endowed with the norm

$$\|v\|_{W^{m,\kappa}(\Omega,\phi)} := \sum_{|\beta| \leq m} \|D^\beta v\|_{L^\kappa(\Omega,\phi)}.$$

We denote by $H^1(\Omega,\phi) = W^{1,2}(\Omega,\phi)$. When $\phi \equiv 1$, these spaces become the usual Sobolev spaces $W^{m,\kappa}(\Omega)$. We denote by $W^{m,\kappa}_{loc}(\Omega)$ the space of functions v such that $v \in W^{m,\kappa}(\Omega')$ for any compact subset $\Omega' \subset \Omega$.

• A Borel measure on Ω is called a Radon measure if it is bounded on compact sets of Ω. Let $\mathfrak{M}(\Omega,\phi)$ be the space of Radon measures τ on Ω satisfying $\int_\Omega \phi\, d|\tau| < \infty$ and $\mathfrak{M}^+(\Omega,\phi)$ be the positive cone of $\mathfrak{M}(\Omega,\phi)$. Denote by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the positive cone of $\mathfrak{M}(\partial\Omega)$. The space $\mathfrak{M}(\Omega,\phi)$ and the space $\mathfrak{M}(\partial\Omega)$ are respectively endowed with the norms

$$\|\tau\|_{\mathfrak{M}(\Omega,\phi)} := \int_\Omega \phi d|\tau|,$$

$$\|\nu\|_{\mathfrak{M}(\partial\Omega)} := \int_{\partial\Omega} d|\nu|.$$

• Denote by $L^\kappa_w(\Omega,\phi)$, $1 \leq \kappa < \infty$, the weak Lebesgue space (or Marcinkiewicz space) with weight ϕ. The subscript w is an abbreviation of "weak". See the definition of weak Lebesgue spaces in subsection 1.3.1.

• Denote $\delta(x) = \mathrm{dist}(x,\partial\Omega)$ where $\partial\Omega$ is the boundary of Ω. When $\phi = \delta^\theta$ with $\theta > -1$, we have the spaces $L^\kappa(\Omega,\delta^\theta)$, $W^{m,\kappa}(\Omega,\delta^\theta)$, $\mathfrak{M}(\Omega,\delta^\theta)$ and $L^\kappa_w(\Omega,\delta^\theta)$.

• A sequence $\{\Omega_n\}$ is a C^2 exhaustion of Ω if $\{\Omega_n\}$ is uniformly of class C^2 and for every n, $\overline{\Omega}_n \subset \Omega_{n+1}$ and $\cup_n \Omega_n = \Omega$.

• Throughout the thesis, c, c_1, c_2, C, C_1, C' denote positive constants which may vary from line to line. We write $C = C(a,b)$ to emphasize the dependence of C on the data a, b.

• The notation $f \sim h$ means that there exist positive constants c_1, c_2 such that $c_1 h < f < c_2 h$.

• For any $a, b \in \mathbb{R}$, we write $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$.

• For $\kappa > 1$, we denote by κ' the conjugate exponent, i.e. $\kappa' = \frac{\kappa}{\kappa-1}$.

• For a set E in $\mathbb{R}^N$, denote by χ_E the indicator function of E.

• For $x \in \mathbb{R}^N$, denote by δ_x the Dirac measure concentrated at x.

• Denote by dS the surface element on $\partial\Omega$.

• For $z \in \partial\Omega$, denote by $\mathbf{n}_z$ the outer unit normal vector at z. We denote by $\frac{\partial}{\partial \mathbf{n}}$ the derivative in the outer normal direction on $\partial\Omega$.

• The gradient of u is $\nabla u = (\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N})$.

1.1.1. Function settings. Nonlinear elliptic equations have been one of the most developed subject in the area of partial differential equations (PDEs) not only because of their great interest to other fields within mathematics such as calculus of variations, harmonic analysis, measure theory, differential geometry, fluid dynamics, probability theory, but also because of their applications in physics, engineering, and other applied scientific disciplines. The simplest second order PDE is the Laplace equation

$$-\Delta u = 0 \quad \text{in } \Omega \tag{1.1.1}$$

where Ω is a domain in the Euclidean space $\mathbb{R}^N$ $(2 \leq N \in \mathbb{N})$ and Δ denotes the Laplace operator (or Laplacian) defined by $\Delta u = \sum_{i=1}^N \frac{\partial^2 u}{\partial x_i^2}$. In (1.1.1) and throughout the present thesis, we write the Laplace operator with 'minus sign' because the operator $-\Delta$ is positive. In the context of this habilitation thesis, unless otherwise stated, Ω is a C^2 bounded domain (see the definition of C^2 domains in Gilbarg and Trugdinger [76]). A function $u \in C^2(\Omega)$ satisfying equation (1.1.1) is called *harmonic*.

Roughly speaking, a boundary value problem for (1.1.1) is a problem of finding an harmonic function u in Ω which satisfies certain auxiliary boundary conditions on some part of the boundary $\partial\Omega$ in some sense. There is a huge literature on boundary value problems for (1.1.1), and for more general elliptic equations, in which one of the earliest well-known works is the Dirichlet problem which asks if we can find an harmonic function u in Ω with a prescribed boundary value $u = h$ on $\partial\Omega$, where h is a given function defined on $\partial\Omega$. The history of such Dirichlet problem is remarkable and led to an extensive development of methods in PDEs in function settings (see a survey by Brezis and Browder [35]).

Another important equation is the Poisson equation which arises in many varied physical situations

$$-\Delta u = \tau \quad \text{in } \Omega, \tag{1.1.2}$$

where τ is a given datum. The Dirichlet problem associated the Poisson equation is

$$\begin{cases} -\Delta u = \tau & \text{in } \Omega, \\ \quad u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1.3}$$

where ν is a given boundary datum. It is classical that if the data τ and ν are smooth enough then problem (1.1.3) admits a unique classical solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ (see for example [76, Theorem 4.3]) and hence the equation and the boundary value condition in (1.1.3) are understood in the pointwise sense. By multiplying the equation in (1.1.3) by $\phi \in C_0^2(\overline{\Omega})$, where

$$C_0^2(\overline{\Omega}) = \{\phi \in C^2(\overline{\Omega}) : \phi = 0 \text{ on } \partial\Omega\}$$

and then using the integration by parts, we obtain the formula

$$-\int_\Omega u\Delta\phi dx = \int_\Omega \tau\phi dx - \int_{\partial\Omega} \nu \frac{\partial \phi}{\partial \mathbf{n}} dS, \tag{1.1.4}$$

where $\mathbf{n}$ is the outer normal unit vector to $\partial\Omega$, $\frac{\partial}{\partial \mathbf{n}}$ denotes the derivative in the outer normal direction on $\partial\Omega$ and dS denotes the surface element on $\partial\Omega$. When τ and ν are not regular, one faces a difficulty stemming from the fact that (1.1.3) does not admit any classical solution. However we observe that functions satisfying (1.1.4) may exist, which leads to the definition of weak solutions. More precisely, Brezis [33] defined weak solutions to (1.1.3) with integrable data as follows:

Assume $\tau \in L^1(\Omega, \delta)$ and $\nu \in L^1(\partial\Omega)$, a function u is a *weak solution* of (1.1.3) if $u \in L^1(\Omega)$ and u satisfies (1.1.4) for all $\phi \in C_0^2(\overline{\Omega})$. Here δ is the distance function to the boundary $\partial\Omega$.

It is known, by the classical approximation method, that for any $\tau \in L^1(\Omega, \delta)$ and $\nu \in L^1(\partial\Omega)$, there exists a unique weak solution u of (1.1.3) (see [116, Proposition 1.1.3]).

Let $G^\Omega : \Omega \times \Omega \setminus \{(x, x) : x \in \Omega\} \to \mathbb{R}_+$ be the Green kernel (or Green function) associated to the operator $-\Delta$ and $P^\Omega : \Omega \times \partial\Omega \to \mathbb{R}_+$ be the Poisson kernel associated to $-\Delta$, i.e.

$$P^\Omega(x, y) = -\frac{\partial G^\Omega}{\partial \mathbf{n}}(x, y) \quad \forall x \in \Omega, y \in \partial\Omega.$$

More properties and sharp estimates of Green kernel and Poisson kernel can be found in [132, 63, 31, 128, 139].

For any $\tau \in L^1(\Omega, \delta)$ and $\nu \in L^1(\partial\Omega)$, the unique weak solution to (1.1.3) can be represented by

$$u(x) = \int_\Omega G^\Omega(x, y)\tau(y)dy + \int_{\partial\Omega} P^\Omega(x, y)\nu(y)dS(y). \tag{1.1.5}$$

The theory of linear problem (1.1.3) forms a basis for the investigation of the Dirichlet problem

$$\begin{cases} -\Delta u + f(u) = \tau & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1.6}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function, τ and ν are given data. The equation in (1.1.6) consists of the linear part Δu and the nonlinear part $f(u)$. This problem has been studied by many authors in various function settings (see, for example, Brezis and Strauss [33, 40], Marcus and Véron [116], Quittner and Souplet [127, 128] and Drábek and Milota [51] and references therein). Weak solutions of problem (1.1.6) are defined Brezis and Strauss as follows:

Assume $\tau \in L^1(\Omega, \delta)$ and $\nu \in L^1(\partial\Omega)$, a function u is a *weak solution* of (1.1.6) if $u \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \delta)$ and u satisfies

$$-\int_\Omega u\Delta\phi dx + \int_\Omega f(u)\phi dx = \int_\Omega \phi\tau dx - \int_{\partial\Omega} \frac{\partial\phi}{\partial \mathbf{n}}\nu dS \quad \forall \phi \in C_0^2(\overline{\Omega}). \tag{1.1.7}$$

Notice that for any $\phi \in C_0^2(\overline{\Omega})$, $|\phi| \le c\delta$, therefore $\phi\tau \in L^1(\Omega)$. Consequently, $L^1(\Omega, \delta)$ is the largest function space for the data.

When $f : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function with $f(0) = 0$ (in this case it is called *absorption nonlinear term*), the solvability of (1.1.6) for any $\tau \in L^1(\Omega, \delta)$ and $\nu \in L^1(\partial\Omega)$ is essentially due to Brezis and Strauss [40] and was demonstrated by Marcus and Véron in [116, Proposition 2.1.2]. It is worth emphasizing that this result holds true for a quite large class of absorption terms since it does not require any additional condition on

f. Therefore, in this regard, linear problem (1.1.3) and nonlinear problem (1.1.6) in L^1 setting share a similarity.

1.1.2. Involvement of measures.

Thomas-Fermi equation. A motivation for the study of semilinear elliptic equations in measure frameworks stems from the Thomas-Fermi theory (see Lieb [95] and Bénilan and Brezis[20]). The theory was invented by L. H. Thomas and E. Fermi in order to describe the electron density $\varrho(x)$, $x \in \mathbb{R}^3$ and the ground state energy for a system (e.g. a molecule) consisting of k nuclei of charges $m_i > 0$ and fixed locations $A_i \in \mathbb{R}^3$ $(1 \leq i \leq k)$ and ℓ electrons. The Thomas-Fermi energy functional for the system is

$$\mathcal{E}(\varrho) = \frac{3}{5}\int_{\mathbb{R}^3} \varrho^{\frac{5}{3}} dx - \int_{\mathbb{R}^3} V\varrho dx + \frac{1}{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{\varrho(x)\varrho(y)}{|x-y|} dxdy + U$$

on

$$\mathcal{A} := \{\varrho \geq 0 : \varrho \in L^1(\mathbb{R}^3) \cap L^{\frac{5}{3}}(\mathbb{R}^3), \int_{\mathbb{R}^3} \varrho dx = \ell\},$$

where $V(x) = \sum_{i=1}^k m_i|x - A_i|^{-1}$ and $U(x) = \sum_{i,j=1}^k m_i m_j |A_i - A_j|^{-1}$. It is noticed that $\varrho \mapsto \mathcal{E}(\varrho)$ is convex. The Thomas-Fermi energy is defined by

$$\mathcal{E}^{TF} := \inf_{\varrho \in \mathcal{A}} \mathcal{E}(\varrho). \tag{1.1.8}$$

The Euler-Lagrange equation, which is also call the Thomas-Fermi equation, is

$$\varrho^{\frac{2}{3}} = (u - \lambda)^+. \tag{1.1.9}$$

where $-\lambda$ is called the Lagrange multiplier or chemical potential and

$$u(x) = V(x) - \int_{\mathbb{R}^3} \frac{\varrho(x)}{|x-y|} dy.$$

It is known that there is a minimizer ϱ for (1.1.8) if and only if $\ell \leq M := \sum_{i=1}^k m_i$. The minimizer is unique, denoted by ϱ^{TF}, and satisfies (1.1.9) for some $\lambda \geq 0$. Conversely, any positive solution of (1.1.9) is a minimizer of (1.1.8). In the neutral case, i.e. $\ell = M$, one has $\varrho > 0$ and $\lambda = 0$, therefore (1.1.9) becomes $\varrho^{\frac{2}{3}} = u$. By applying Δ on both sides, one obtains a semilinear equation

$$-\Delta u + 4\pi\, u^{\frac{3}{2}} = 4\pi \sum_{i=1}^k m_i\, \delta_{A_i} \tag{1.1.10}$$

where δ_{A_i} denotes the Dirac measure concentrated at A_i. It can be seen that the left-hand side of this equation consists of a linear part Δu and a nonlinear part which is expressed by $u^{\frac{3}{2}}$, while the right-hand side is the sum of Dirac measures concentrated at the points A_i.

Interior measure data. Motivated by the investigation on equation (1.1.10), Bénilan and Brezis [20] considered a more general equation

$$-\Delta u + |u|^{p-1}u = \tau \quad \text{in } \Omega \tag{1.1.11}$$

where Ω is a domain in $\mathbb{R}^N$, $p > 1$ and τ is a Radon measure on Ω. Equation (1.1.10) is a particular case of (1.1.11) with $N = 3$ and $p = \frac{3}{2}$. The analysis of this equation reveals that the theory in L^1 setting previously established by Brezis and Strauss [40] cannot be easily extended to a measure framework and new striking phenomena appear due to the nonlinear nature of the problem. Therefore, dealing with a measure framework would provide a deep insight about the problem. In particular, Bénilan and Brezis [20] showed that the value $\frac{N}{N-2}$ is a *critical exponent* for the existence of equation (1.1.11) with zero Dirichlet boundary condition $u = 0$. More precisely, they proved that if $1 < p < \frac{N}{N-2}$ then the problem has a unique solution, while if $p \geq \frac{N}{N-2}$ then there is no solution with $\mu = \delta_A$ for any $A \in \Omega$. Many developments have been achieved since the work of Bénilan and Brezis [20], including Véron [136] and Brezis and Oswald [39] for a complete classification of solutions with an isolated singularity, Vázquez and Véron [134, 135] for more general nonlinearities, Brezis and Véron [41] for removable isolated singularities, Baras and Pierre [14, 15] for removability results in terms of Bessel capacities, Vázquez and Véron [133] and Friedman and Véron [70] for isolated singularities of quasilinear equations.

Boundary measure data. Similar problems with boundary measures have been also studied with important motivation coming from the probability theory. Boundary value problems with measure data for linear and semilinear equations are respectively related to Markov processes called diffusions and superdiffusions. A diffusion is a model of a random motion of a single particle and is characterized by a second order elliptic differential operator, including the Laplacian. A superdiffusion, which describes a random evolution of a cloud of particles, is closely related to semilinear equations. For further discussions about the importance of measure boundary data in the study of linear and semilinear equations in connection with diffusions and superdiffusions, the reader is referred to excellent books of Dynkin [57, 58].

The role of boundary measures can be seen in particular from the representation theorem for harmonic function. More precisely, given a positive harmonic function in Ω, by Herglotz-Doob theorem [116, theorem 1.4.1], there exists a unique measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$u(x) = \int_{\partial\Omega} P^{\Omega}(x,y)d\nu(y). \tag{1.1.12}$$

holds. Such measure ν is called *boundary measure* of u and it is attained as the limit of the Sobolev trace of the solution u in each surface parallel to $\partial\Omega$. More precisely, let $\{\Omega_n\}$ be a C^2 exhaustion of Ω and denote by $u|_{\partial\Omega_n}$ the Sobolev boundary trace of u on $\partial\Omega_n$. Then there exists a nonnegative bounded Radon measure ν on $\partial\Omega$ independent of the choice of the exhaustion such that the sequence of measures $\{u|_{\partial\Omega_n}dS\}$ converges weakly to ν. The above result shows that in order to completely characterize the boundary behavior of harmonic functions, it is insufficient to deal only with function settings and hence measures have to be involved in the analysis.

1.1.3. Measure frameworks.

Linear equations. The results for linear equations in function settings can be extended to measure frameworks in which the definition of weak solutions to (1.1.3) is modified as follows:

Assume $\tau \in \mathfrak{M}(\Omega, \delta)$ and $\nu \in \mathfrak{M}(\partial\Omega)$. A function u of (1.1.3) is a solution of (1.1.3) if

$$- \int_\Omega u\Delta\phi dx = \int_\Omega \phi d\tau - \int_{\partial\Omega} \frac{\partial \phi}{\partial \mathbf{n}} d\nu, \quad \forall \phi \in C_0^2(\overline{\Omega}). \tag{1.1.13}$$

As explained in L^1 setting, for any $\phi \in C_0^2(\overline{\Omega})$, one has $|\phi| \leq c\delta$, and hence the first term on the right-hand side of (1.1.13) is finite. This also explains why $\mathfrak{M}(\Omega, \delta)$ is the largest possible measure space one can work on. The second term on the right-hand side is finite because $\frac{\partial \phi}{\partial \mathbf{n}}$ is bounded.

It is known that, for any $\tau \in \mathfrak{M}(\Omega, \delta)$ and $\nu \in \mathfrak{M}(\partial\Omega)$, there exists a unique solution of (1.1.3) and the solution can be represented by the Green kernel acting on τ and the Poisson kernel acting on ν (see [**116**, Theorem 1.2.2]), i.e.

$$u(x) = \int_\Omega G^\Omega(x, y) d\tau(y) + \int_{\partial\Omega} P^\Omega(x, y) d\nu(y). \tag{1.1.14}$$

In particular, given a measure $\nu \in \mathfrak{M}(\partial\Omega)$, the harmonic solution with boundary condition $u = \nu$ is given by

$$u(x) = \int_{\partial\Omega} P^\Omega(x, y) d\nu(y). \tag{1.1.15}$$

Absorption nonlinearities. Over the last decades, boundary value problems for semilinear equations in measure frameworks have been intensively investigated both in probabilistic approaches and analytic methods with the aim of bringing into light and describing several aspects of nonlinear phenomena. The pioneering work on the Dirichlet problem with measure boundary data for semilinear elliptic equations with an *absorption term*

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega, \tag{1.1.16}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing, continuous function with $f(0) = 0$, is due to Gmira and Véron [**82**]. They introduced the definition of weak solutions of

$$\begin{cases} -\Delta u + f(u) = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1.17}$$

in spirit of Brezis [**33**] as follows:

Assume $\nu \in \mathfrak{M}(\partial\Omega)$. A function u is a weak solution of (1.1.17) if $u \in L^1(\Omega)$, $f(u) \in L^1(\Omega, \delta)$ and u satisfies

$$- \int_\Omega u\Delta\phi dx + \int_\Omega f(u)\phi dx = - \int_{\partial\Omega} \frac{\partial \phi}{\partial \mathbf{n}} d\nu \quad \forall \phi \in C_0^2(\overline{\Omega}). \tag{1.1.18}$$

The notion of weak solutions is well defined. Indeed, for any $\phi \in C_0^2(\overline{\Omega})$, one has $\Delta\phi \in L^\infty(\Omega)$ and hence $u\Delta\phi \in L^1(\Omega)$. Therefore the first term on the left-hand side of (1.1.18) is finite. The second term on the left-hand side of (1.1.18) is also finite due to the fact that $|\phi| \leq c\delta$ and $f(u) \in L^1(\Omega, \delta)$. It can be also seen that the term on the right-hand side of (1.1.18) is also finite because $\frac{\partial \phi}{\partial \mathbf{n}}$ is bounded on $\partial\Omega$ and ν is a bounded Radon measure on $\partial\Omega$.

A highlighting feature is that, in contrast to the L^1 case where the existence holds true for every L^1 boundary datum, the Dirichlet problem

(1.1.17) is not solvable for every measure datum in general. More precisely, Gmira and Véron [82] showed that if f satisfies

$$\int_1^\infty (f(t) + |f(-t)|) t^{-1-\frac{N+1}{N-1}} dt < \infty, \tag{1.1.19}$$

then problem (1.1.17) has a unique weak solution. This result was demonstrated by employing a classical approximation method, in combination with Marcinkiewicz estimates on Green kernel and Martin kernel and Vitali convergence theorem. Condition (1.1.19) is sharp and the value $\frac{N+1}{N-1}$ appearing in (1.1.19) is a *critical exponent* for the existence of a solution to (1.1.17). When $f(u) = |u|^{p-1}u$ with $p > 1$, equation (1.1.16) becomes

$$-\Delta u + |u|^{p-1}u = 0 \quad \text{in } \Omega, \tag{1.1.20}$$

and condition (1.1.19) is interpreted as $1 < p < \frac{N+1}{N-1}$, which is the *subcritical range*. Hence in this range, for any $\nu \in \mathfrak{M}(\partial\Omega)$, problem

$$\begin{cases} -\Delta u + |u|^{p-1}u = 0 & \text{in } \Omega, \\ \qquad\qquad\quad u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1.21}$$

admits a unique weak solution in this case. In the *supercritical range*, namely $p \geq \frac{N+1}{N-1}$, it was shown (see [82]) that there is no weak solution of (1.1.21) if ν is a Dirac measure concentrated at a point on $\partial\Omega$. Furthermore, when $p \geq \frac{N+1}{N-1}$, any nonnegative solution $u \in C(\overline{\Omega} \setminus \{0\})$ of (1.1.20) vanishing on $\partial\Omega \setminus \{0\}$ is identically zero, i.e. *isolated boundary singularities are removable.*

The topic has reached to its full flowering through a series of celebrated papers by Marcus and Véron [109, 111, 112, 113, 114, 115, 99, 116] and many other works (see for examples [37, 30, 120, 29, 22] and references therein). Taking into account the construction of the boundary measure of positive harmonic functions, Marcus and Véron introduced a notion of boundary trace [116, Definition 1.3.6] in order to describe the boundary behavior of solutions to equation (1.1.16).

Definition 1.1 (m-boundary trace). Let $u \in W^{1,\kappa}_{loc}(\Omega)$ for some $\kappa > 1$. We say that u possesses an *m-boundary trace* on $\partial\Omega$ if there exists a bounded Radon measure ν on $\partial\Omega$ such that, for every C^2 exhaustion $\{\Omega_n\}$ of Ω and every $\varphi \in C(\overline{\Omega})$,

$$\lim_{n\to\infty} \int_{\partial\Omega_n} u|_{\partial\Omega_n} \varphi dS = \int_{\partial\Omega} \varphi d\nu.$$

Here $u|_{\partial\Omega_n}$ denotes the Sobolev trace of u on $\partial\Omega_n$. The m-boundary trace of u is denoted by $\mathrm{tr}(u)$.

It is known from [116, Proposition 1.3.7] that every positive harmonic function u in Ω admits a positive m-boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$, which in fact coincides the boundary measure given by (1.1.15).

A first step to study the notion of m-boundary trace is to deal with *moderate solutions* of (1.1.16), namely weak solutions of (1.1.16) which are bounded by positive harmonic functions (see [116, Definition 3.1.1]). The equivalence between the notion of m-boundary trace and the concept of moderate solutions was established by Marcus and Véron [116, Theorem 3.1.2]. Moreover, these notions are equivalent to the condition $f(u) \in L^1(\Omega, \delta)$.

Afterwards, Marcus and Véron generalized the notion of m-boundary trace to a notion called *rough boundary trace* and pointed out that, under some additional assumptions on f, every positive solution of (1.1.16) possesses a rough boundary trace given by a positive outer regular Borel measure on $\partial\Omega$ (see [115, Theorem 1.1] or [116, Theorem 3.1.8, Theorem 3.1.12 and Definition 3.1.14]). Conversely, they showed that the solvability for (1.1.17) holds true when ν is a positive outer regular Borel measure which may be infinite on subsets of $\partial\Omega$ (see [116, Theorem 3.3.1]). In particular, if $f(u) = |u|^{p-1}u$ then, in the subcritical range $p < \frac{N+1}{N-1}$, for any positive outer regular Borel measure ν, problem (1.1.21) admits a unique positive solution (see [115, Theorem 1.6]). Also, in the subcritical range, Marcus and Véron [111, 115] characterized completely boundary isolated singularities of nonnegative solutions of (1.1.20). It means that if $u \in C(\overline{\Omega} \setminus \{0\})$ is a nonnegative solution of (1.1.20) vanishing on $\partial\Omega \setminus \{0\}$, then either $u = u_k$, the solution of (1.1.21) with $\nu = k\delta_0$ for some $k \geq 0$ (*weakly singular solutions*), or $u = \lim_{k\to\infty} u_k$ (*strongly singular solution*).

The supercritical case is more challenging and was treated by many authors using various techniques. The removability result due to Gmira-Véron has been significantly extended, either by using probabilistic approach by Le Gall [92], [93], Dynkin and Kuznetsov [59], [60], under the restriction $\frac{N+1}{N-1} \leq p \leq 2$, or by employing purely analytic methods by Marcus and Véron [112, 113, 114] in the whole range $\frac{N+1}{N-1} \leq p$. The key ingredient in analyzing the problem is the Bessel capacity $C_{\frac{2}{p},p'}$ in $(N-1)$-dimension, where $p' = \frac{p}{p-1}$. Among the most interesting results, it is worth mentioning that problem (1.1.21) is solvable with $\nu \in \mathfrak{M}^+(\partial\Omega)$ if and only if ν is absolutely continuous with respect to the $C_{\frac{2}{p},p'}$-capacity. Furthermore, if $E \subset \partial\Omega$ is compact and $u \in C(\overline{\Omega} \setminus E)$ is a solution of (1.1.20) vanishing on $\partial\Omega \setminus E$, then u is necessary zero if and only if $C_{\frac{2}{p},p'}(E) = 0$. A complete characterization of positive solutions of (1.1.20) has been developed by Mselati [117] when $p = 2$, by Dynkin [57, 58] for $\frac{N+1}{N-1} \leq p \leq 2$, and finally by Marcus [99] for the whole supercritical range $p \geq \frac{N+1}{N-1}$.

Source nonlinearities. An important PDE with a source term is the Lane-Endem equation

$$-\Delta u = u^p \quad \text{in } \Omega, \tag{1.1.22}$$

where $p > 1$, which was named after astrophysicists Jonathan Homer Lane and Robert Emden. This equation was introduced in 1869 by Home Lane [89] in the study of the temperature and the density of mass on the surface of the Sun and has received special attention because it can be used to describes polytropes in hydrostatic equilibrium as simple models of a star. A systematic survey on the this equation from the physical and mathematical point of view was presented in [42, 17]. A great number of remarkable works have been carried out in various directions by many mathematicians, including Lions [96], Brezis and Nirenberg [38], Gidas, Ni and Nirenberg [73], Gidas and Spruck [74, 75], Baras and Pierre [14, 15] and Kalton and Verbitsky [86], Poláčik, Quittner and Souplet [125], Quittner and Souplet [127, 128].

It is worth mentioning that a universal pointwise estimate for nonnegative solutions of (1.1.22) was obtained by Poláčik, Quittner and Souplet [125, Theorem 2.3] for $1 < p < \frac{N+2}{N-2}$ by using a rescaling argument, a key doubling property and Liouville-type results.

The Dirichlet problem with measure boundary data

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.1.23}$$

was first studied by Bidault-Véron and Vivier in [31] where estimates involving classical Green and Poisson kernels for the Laplacian were established to obtain an existence result in the subcritical range, i.e. $1 < p < \frac{N+1}{N-1}$. Afterwards, Bidault-Véron and Yarur [32] reconsidered this type of problem in a more general setting and gave a necessary and sufficient condition for the existence of solutions. Chen et al. [43] investigated the Dirichlet problem with a more general source term by using the Schauder fixed point theorem in combination with weighted Marcinkiewicz estimates. Recently, Bidaut-Véron et al. [28] provided new criteria expressed in terms of boundary capacities for the existence of weak solutions to problem (1.1.23).

A remarkable feature of the Dirichlet problem (1.1.23) is that, not only the value of exponent p, but also the total mass of the boundary datum ν plays an important role in deriving the existence and non-existence result. More precisely, when $1 < p < \frac{N+1}{N-1}$, there exists a threshold value ρ^* such that problem (1.1.23) admits a solution if $\|\nu\|_{\mathfrak{M}(\partial\Omega)} \leq \rho^*$, and no solution if $\|\nu\|_{\mathfrak{M}(\partial\Omega)} > \rho^*$ (see [31]).

It is worth mentioning that existence results for scalar equations were extended to the Lane-Emden system

$$\begin{cases} -\Delta u = v^p + \tau & \text{in } \Omega, \\ -\Delta v = u^{\tilde{p}} + \tilde{\tau} & \text{in } \Omega, \\ u = \nu, \quad v = \tilde{\nu} & \text{on } \partial\Omega, \end{cases} \tag{1.1.24}$$

where $\tau, \tilde{\tau}$ are measures in Ω and $\nu, \tilde{\nu}$ are measures on $\partial\Omega$. Various existence and non-existence results, as well as a priori estimates, for solutions of (1.1.24) were established in [32]. See also the celebrated paper of Poláčik, Quittner and Souplet [125] and the books of Quittner and Souplet [127, 128] for related results.

Gradient-dependent nonlinearities. Equations with a nonlinear term depending on the gradient of solutions (or the convection) arise in various models of optimal stochastic control with state constrains. A typical equation is

$$-\Delta u + |\nabla u|^q = 0 \quad \text{in } \Omega, \tag{1.1.25}$$

with $q > 1$, which is also a particular case of Hamilton-Jacobi-Bellman equations. Existence, a priori estimates and qualitative properties of solutions to this equation, as well as more general class of equations, were discussed in Lions [97] and Lasry and Lions [90]. The Dirichlet problem with measure data

$$\begin{cases} -\Delta u + f(|\nabla u|) = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega \end{cases} \tag{1.1.26}$$

where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function with $f(0) = 0$, was first studied in our joint work with Véron [120]. Under the subcriticality integral condition on f given by

$$\int_1^\infty f(t)t^{-1-\frac{N+1}{N}}dt < +\infty,$$

we obtained the existence of a positive solution to (1.1.26). When f is of power type, namely $f(|\nabla u|) = |\nabla u|^q$ with $1 < q < 2$, the subcriticality integral condition reads as $1 < q < \frac{N+1}{N}$. It was showed that $\frac{N+1}{N}$ is the critical exponent for the solvability of (1.1.26) and a complete description of the structure of solutions with an isolated singularity on $\partial\Omega$ was provided. Moreover, the existence of a solution to the Dirichlet problem with boundary datum given by a Borel measure in the subcritical case, i.e. $1 < q < \frac{N+1}{N}$, and a removability result in the supercritical case, i.e. $\frac{N+1}{N} \le q < 2$, were also established.

These results were then extended in our joint paper with Marcus [105] to the Dirichlet problem for a much more intricate equations with a nonlinear term depending on both solutions and their gradient

$$\begin{cases} -\Delta u + f(u, |\nabla u|) = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega \end{cases} \tag{1.1.27}$$

Two model cases $f(u, |\nabla u|) = u^p + |\nabla u|^q$ and $f(u, |\nabla u|) = u^p|\nabla u|^q$ were carefully studied in [105].

It is worth mentioning that equations with a gradient-dependent nonlinear term have been studied in various directions. We refer to Ghergu and Rădulescu [71, 72] for singular elliptic equations with convection term and zero Dirichlet condition, Alarcón, García-Melián [5, 6] for Keller-Osserman type estimates and Liouville type theorems, Aghajani, Cowan and Lui [2, 3, 4] for singular solutions, Bidaut-Véron, Garcia-Huidobro and Véron [25, 26, 27] for a priori estimates on singular solutions.

1.2. Elliptic equations with a Hardy potential

In this section, we first explain the role of Hardy potentials and measures in the investigation. Then we address the main problems regarding elliptic equations with a Hardy potential in the thesis and depict briefly the main results collected from our papers [106, 119, 78, 79, 80]. The detailed statements of these results will be provided in Section 1.3.

1.2.1. The role of Hardy potentials. Schrödinger operators of the form $L_V = \Delta + V$, where V is a potential, have been intensively investigated by numerous mathematicians because of their applications in nonrelativistic quantum mechanics, geometry, spectral and scattering theory, and integrable systems (see for example [129, 131]). The behavior of the potential V has a significant effect on properties of L_V such as spectral properties and the existence of the associated Green kernel and Martin kernel.

The Coulombian potential $V(x) = \mu|x|^{-1}$, $\mu \in \mathbb{R}$, appears in the Thomas-Fermi-Dirac-von Weizsäcker theory [18, 19]. The case where V is an inverse square potential, i.e. $V(x) = \mu|x|^{-2}$, is called Leray-Hardy potential and has been studied in connection with semilinear elliptic equations by Guerch and Véron [83], Cîrstea [47], Dupaigne [55], Dávila and Dupaigne [49, 50], Du and Wei [53], Chen and Véron [44].

When singular potentials, which blow up on the boundary $\partial\Omega$, were involved, in most of the cases, they were assumed to satisfy, e.g. $|V(x)| \leq c\delta(x)^{2-\epsilon}$ for some $\epsilon > 0$ small or more generally $\delta V \in L^1(\Omega)$ (see [141, 122, 123]), which exclude the case where V behaves like δ^{-2} on certain part of Ω. The case where

$$|V(x)| \leq C\delta(x)^{-2} \tag{1.2.1}$$

is of great interest and a theory of linear Schrödinger equation $L_V u = 0$ on manifolds was successfully and systematically developed by Ancona [7, 8]. The particular potential $V(x) = \mu\delta(x)^{-2}$ has received special attention and it is called Hardy potential because of the close link to the Hardy inequality

$$C_H(\Omega) \int_\Omega \frac{|\varphi|^2}{\delta^2} dx \leq \int_\Omega |\nabla\varphi|^2 dx \quad \forall \varphi \in H_0^1(\Omega), \tag{1.2.2}$$

where $C_H(\Omega)$ denotes the best constant in Hardy inequality (also called Hardy constant) given by

$$C_H(\Omega) := \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla\varphi|^2 dx}{\int_\Omega (\varphi/\delta)^2 dx}. \tag{1.2.3}$$

In spirit of [69], Hardy inequality (1.2.2) can be interpreted as a form of *uncertainty principle*. This means if the function φ in (1.2.2) is localized close to the boundary (i.e. the term on the left-hand side of (1.2.2) is large) then its momentum becomes large (i.e. the left-hand side of (1.2.2) is large). The exponent -2 appearing in the power δ^{-2} plays a crucial role because it keeps inequality (1.2.2) scaling invariant.

Inequality (1.2.2) in one dimensional case with Hardy constant $C_H(\Omega) = \frac{1}{4}$ was discovered by Hardy [84, 85]. Moreover, he pointed out that the Hardy constant is not attained. This inequality was then extended to Lipschitz domains in $\mathbb{R}^N$ by Necas [118], Opic and Kufner [121] and was revisited by Brezis and Marcus [36]. It is classical that $C_H(\Omega) \in (0, \frac{1}{4}]$ and $C_H(\Omega) = \frac{1}{4}$ if Ω is convex (see [103, Theorem 11]) or if $-\Delta\delta \geq 0$ in the sense of distributions (see [16, Theorem A]). Moreover, the infimum in (1.2.3) is achieved if and only if $C_H(\Omega) < \frac{1}{4}$.

For $\mu \in \mathbb{R}$, denote by L_μ the Laplace operator perturbed by a Hardy potential

$$L_\mu := \Delta + \frac{\mu}{\delta^2}, \tag{1.2.4}$$

where $\delta(x) = \mathrm{dist}(x, \partial\Omega)$. Notice that the Hardy potential blows up on $\partial\Omega$. The energy functional associated to the operator L_μ is given by

$$E_\mu(\varphi) = \frac{1}{2} \int_\Omega \left(|\nabla\varphi|^2 - \frac{\mu}{\delta^2}\varphi^2 \right) dx, \quad \varphi \in H_0^1(\Omega). \tag{1.2.5}$$

We see that if $\mu \leq C_H(\Omega) (\leq \frac{1}{4})$ then the energy functional is bounded from below and important spectral properties of $-L_\mu$ can be derived, which plays an important role in the study of linear and semilinear equations involving $-L_\mu$ in a variational framework.

Therefore, the first and important step in the investigation of boundary value problem for equations involving Schrödinger operators L_V satisfying (1.2.1) is to understand the operator L_μ with the restriction $\mu \leq \frac{1}{4}$.

1.2.2. The role of measures. Similarly to the free potential case, i.e. $\mu = 0$, a heuristic strategy in the study of linear and nonlinear equations involving Hardy potentials is first to treat them in a function setting and then to extend the scope of the theory to a measure framework.

In function settings, the Dirichlet problems for linear and nonlinear equations involving Hardy potentials share a similarity. More precisely, the linear problem admits a unique solution for any L^1 data and nonlinear problem is solvable for any L^1 data with a large class of nonlinear terms. This can be seen in [**77**, Proposition 3.2] and [**106**, Corollary C1].

However, beyond the similarity, there are sharp distinctions between linear phenomena and nonlinear phenomena which have not been well understood until the involvement of measures in the analysis. The extension from a function setting to a measure framework appears to be an appropriate approach to reveal and interpret these distinctions. The first one is that, in contrast to the linear case where the solvability is valid for any bounded Radon measure data (see [**106**, Proposition I]), in the nonlinear case, the existence depends essentially on the nonlinear term and/or the total mass of the boundary data and it is possible to construct bounded measures (for example Dirac measures) for which the nonexistence occurs (see [**106**, Theorem F]). Moreover, the multiplicity result may hold if the total mass of the boundary data is small enough. The second distinction is that, unlike the linear case where any positive solution of homogeneous linear equations can be uniquely represented via a positive bounded measure on $\partial\Omega$, in the nonlinear case, there are positive solutions of nonlinear equations whose boundary behavior is given by a Borel measure which may take infinite values on subsets of $\partial\Omega$. These results will be discussed in the next subsections (see also [**106, 119, 77**]).

1.2.3. Introduction of main problems. In this thesis, we are interested in boundary value problems for nonlinear elliptic equations involving operator L_μ (defined in (1.2.4)) of the form

$$-L_\mu u \pm g(u, |\nabla u|) = 0 \quad \text{in } \Omega, \tag{1.2.6}$$

where g is nondecreasing with respect to u and/or $|\nabla u|$. These equations consist of two competing effects: the diffusion driven by the linear operator L_μ and the reaction term expressed by the nonlinear term $g(u, |\nabla u|)$. We will focus on the range $\mu \in (0, \frac{1}{4}]$ in which interesting properties of $-L_\mu$ are exploitable (see the explanation in subsection 1.3.1). The nonlinear term $g(u, |\nabla u|)$ is called *absorption* (resp. *source*) if the 'plus sign' (resp. 'minus sign') appears in (1.2.6). Typical models are $g(u) = |u|^{p-1}u$, $g(|\nabla u|) = |\nabla u|^q$, $g(u, |\nabla u|) = u^p + |\nabla u|^q$ and $g(u, |\nabla u|) = u^p |\nabla u|^q$.

Aim. The aim of the thesis is to present recent developments on nonlinear equation (1.2.6). In particular, the following problems are addressed.

(1) The boundary trace problem: We aim to show that any positive solution u of (1.2.6) can be uniquely characterized by a positive measure on

the boundary $\partial\Omega$. Roughly speaking, such a measure, if exists, is called the boundary trace of u. This result is not only an extension of the Representation theorem for harmonic functions to solutions of nonlinear equations, but also reveal new phenomena in the nonlinear case.

(2) The Dirichlet problem with measures for (1.2.6): Given a measure ν defined on the boundary $\partial\Omega$, we look for a solution to equation (1.2.6) satisfying boundary condition $u = \nu$ in some sense. We also discuss various necessary and sufficient conditions for the existence of solutions to (1.2.6). Furthermore, we consider the questions of non-existence, uniqueness and multiplicity in particular cases.

(3) Isolated boundary singularities: We investigate the case where the boundary data concentrate only at a point on the boundary. More precisely, we aim to give a complete description of the set of solutions to equation (1.2.6) vanishing on $\partial\Omega \setminus \{y\}$ for $y \in \partial\Omega$. This result will provide the exact asymptotic behavior of such solutions near the isolated boundary singularities.

(4) Removable boundary singularities: Assume $F \subset \partial\Omega$ and $u \in C(\overline{\Omega}\setminus F)$ is a nonnegative solution of (1.2.6) in Ω vanishing on $\partial\Omega\setminus F$. We will look for conditions on the set F and the nonlinear term g under which the 'singular set' F can be removable, i.e. u is identically zero. In particular, this result shows the nonexistence of singular solutions when the nonlinear term grows 'fast'.

Features. The above problems have the following main features.

• The presence of the Hardy potential which blows up on the boundary $\partial\Omega$ has an essential effect on the boundary behavior of solutions to (1.2.6). Consequently, the boundary conditions cannot be imposed arbitrarily and the Dirichlet problem for (1.2.6) cannot be handled by the classical techniques.

• In general, universal estimates (for example the Keller-Osserman estimate [**87**]) do not hold for positive solutions of (1.2.6).

• The expression of the nonlinear term g plays a crucial role in the study of the solvability for boundary value problems for (1.2.6). In particular, the absorption case and the source case are sharply different in the sense that in the absorption case, the existence and uniqueness do not depend on the total mass of the boundary data, while in the source case, the existence relies not only the concentration but also on the total mass of the boundary data and the uniqueness breaks down.

• When the nonlinear term g depends on the gradient of solutions, equation (1.2.6) becomes *non-variational*, it means that this equation cannot be solved by using variational methods. Furthermore, the dependence of g on the gradient ∇u causes the *lack of monotonicity*, which can be seen from an easy observation that the inequality $u(x) \le v(x)$ does not imply any relation between $|\nabla u(x)|$ and $|\nabla v(x)|$; therefore approaches based on the monotonicity are invalid. In addition, the competition between u^p and $|\nabla u|^q$ generates the complication of the study.

• In general, equation (1.2.6) is not *scaling invariant* (or in other words, equation (1.2.6) does not admit any *similarity transformation*), i.e. if u is a solution of (1.2.6) then the function $v(x) = \ell^\alpha u(\ell^\beta x)$, for $\ell > 0$, $\alpha, \beta \in \mathbb{R}$, does not solve (1.2.6). Consequently, standard arguments relying on the scaling invariance property are not applicable.

• The boundary data are given by measures on the boundary $\partial\Omega$, which makes solutions (if exist) significantly less regular. Therefore, the standard approximation arguments are invalid or may be valid only under some additional conditions on the nonlinear term g.

The interaction of the above features yields substantial difficulties and leads to disclose new types of results. Therefore a new approach with subtle analysis is required in the investigation.

1.2.4. Brief description of our contributions. This subsection serves as a summary of the main results in the thesis and provides a comparison with previous results. This might help the reader grasp the gist of the main results. The reader who is interested in the detailed statements and the proofs is referred to Section 1.3 and Chapters 2–5.

Boundary trace problem. We consider the boundary trace problem for equation (1.2.6). To this purpose, we first investigate the boundary behavior of *L_μ-harmonic functions* in Ω, i.e. solutions of the equation

$$-L_\mu u = 0 \quad \text{in } \Omega, \tag{1.2.7}$$

which in turn indicates possible boundary behavior of solutions to corresponding semilinear equations. Based on that, we introduce a notion of *normalized μ-boundary trace* expressed by a bounded measure on the boundary (see Definition 1.3). This notion is new (compared with the notion of m-boundary trace when $\mu = 0$ in Definition 1.1) because it depicts clearly how a function admitting a normalized μ-boundary trace behaves on every surface parallel to the boundary $\partial\Omega$. Moreover, it is pointed out that this notion is equivalent to the concept of *moderate solutions* of (1.2.6), namely solutions which are dominated by a positive L_μ-harmonic function. In parallel, another notion of boundary trace defined by mean of weak convergence of measures is introduced by Gkikas and Véron [77]. We then show that these notions of boundary trace are equivalent (see subsection 1.3.2). Furthermore, it is worth mentioning that any positive solution (not necessarily moderate) admits a boundary trace given by a Borel measure on $\partial\Omega$ which may be infinite on compact subsets of $\partial\Omega$ (see [77, Theorem F]).

Dirichlet problem. We investigate the Dirichlet problem

$$\begin{cases} -L_\mu u \pm g(u, |\nabla u|) = 0 & \text{in } \Omega, \\ u = \nu & \text{on } \partial\Omega, \end{cases} \tag{1.2.8}$$

where ν is a given measure on $\partial\Omega$, $\mu \in (0, \frac{1}{4}]$ and g is a nondecreasing function with respect to u and/or $|\nabla u|$. Since the analysis depends essentially on the expression of the nonlinear term, typical models are considered separately, for which we showed that if the nonlinear term g does not grow 'too fast' with respect to u and/or $|\nabla u|$, then for any bounded Radon measure ν on $\partial\Omega$, problem (1.2.8) possesses a solution.

Absorption case. Let us illustrate the above-mentioned fact by considering the absorption case, namely the equation (1.2.8) with plus sign. The following exponent is deeply involved in the analysis

$$\alpha := \frac{1}{2}(1+\sqrt{1-4\mu}). \tag{1.2.9}$$

This exponent comes from the construction of a special solution of (1.2.7) in the half space $\mathbb{R}^N_+$ which is singular at the origin 0 and vanishes on $\partial\mathbb{R}^N_+ \setminus \{0\}$ (Actually this solution plays a similar role as the Poisson kernel of $-\Delta$ in the half space). Notice that, since $\mu \in (0, \frac{1}{4}]$, $\alpha \in [\frac{1}{2}, 1)$.

When $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing, continuous function depending only on solutions u with $g(0) = 0$, we show that the condition

$$\int_1^\infty (g(t) + |g(-t)|)t^{-1-p_\mu} dt < \infty, \tag{1.2.10}$$

with

$$p_\mu := \frac{N+\alpha}{N+\alpha-2}, \tag{1.2.11}$$

is a sufficient condition for the existence of (1.2.8) (see Theorem 1.10 and [77, Theorem 3.3]). A typical model is $g(u) = |u|^{p-1}u$ with $p > 1$ and (1.2.10) is satisfied if and only if $1 < p < p_\mu$. For this model, p_μ is called a *critical exponent* for (1.2.8). We say that p is in the *subcritical range* if $p < p_\mu$, otherwise we say that p is in the *supercritical range*.

When $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous function depending only on $|\nabla u|$ with $g(0) = 0$, we show that the condition

$$\int_1^\infty g(t)t^{-1-q_\mu} dt < \infty, \tag{1.2.12}$$

with

$$q_\mu := \frac{N+\alpha}{N+\alpha-1}, \tag{1.2.13}$$

is a sufficient condition for the existence of (1.2.8) (see [79, Theorem B]). A typical model is $g(|\nabla u|) = |\nabla u|^q$ with $1 < q < 2$ and (1.2.12) is satisfied if and only if $1 < q < q_\mu$. For this model, q_μ is called a *critical exponent* for (1.2.8). We say that q is in the *subcritical range* if $q < q_\mu$, otherwise we say that q is in the *supercritical range*.

When $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing, continuous function depending on both solutions u and the gradient $|\nabla u|$ with $g(0,0) = 0$, we point out that (see [80, Theorem 1.3]) the sufficient condition for the existence of (1.2.8) can be nicely expressed by a combination of condition (1.2.10) and condition (1.2.12) as

$$\int_1^\infty g(t, t^{\frac{p_\mu}{q_\mu}})t^{-1-p_\mu} dt < \infty. \tag{1.2.14}$$

There are two typical models that are worth highlighting.

• The first one is $g(u, |\nabla u|) = u^p + |\nabla u|^q$ with $p > 1$ and $1 < q < 2$ and (1.2.14) is satisfied if

$$1 < p < p_\mu \quad \text{and} \quad 1 < q < q_\mu. \tag{1.2.15}$$

For this model, we say that (p, q) is in the *subcritical range* if (1.2.15) holds, otherwise we say that (p, q) is in the *supercritical range*.

• The second model is $g(u,|\nabla u|)=u^p|\nabla u|^q$ with $p\ge 0, 0\le q<2$ and $p+q>1$ and (1.2.14) is satisfied if

$$(N+\alpha-2)p+(N+\alpha-1)q<N+\alpha. \tag{1.2.16}$$

For this model, we say that (p,q) is in the *subcritical range* if (1.2.16) holds, otherwise we say that (p,q) is in the *supercritical range*.

Sharp solvability results in the typical models can be summarized in the following table [1].

Absorption case	Existence, uniqueness in subcritical range	Non-existence in supercritical range
$g(u)=\lvert u\rvert^{p-1}u$	$p<p_\mu$	$p\ge p_\mu$
$g(\lvert\nabla u\rvert)=\lvert\nabla u\rvert^q$	$q<q_\mu$	$q_\mu\le q<2$
$g(u,\lvert\nabla u\rvert)=u^p+\lvert\nabla u\rvert^q$	$p<p_\mu$ and $q<q_\mu$	$p\ge p_\mu$ or $q_\mu\le q<2$
$g(u,\lvert\nabla u\rvert)=u^p\lvert\nabla u\rvert^q$	$(N+\alpha-2)p+$ $(N+\alpha-1)q<N+\alpha$	$(N+\alpha-2)p+$ $(N+\alpha-2)p\ge N+\alpha$

Table 1: Absorption case

The above table shows that, in the subcritical range, *for any bounded boundary measure* ν, problem (1.2.8) with plus sign admits a *unique* solution. Moreover, this solution is bounded from above by the solution to the corresponding linear problem

$$\begin{cases} -L_\mu u=0 & \text{in } \Omega,\\ \quad\; u=\nu & \text{on } \partial\Omega. \end{cases} \tag{1.2.17}$$

The proof of the existence result relies on the approximation method, making use of sub and super solutions theorem, delicate estimates of Green kernel and Martin kernel in weak Lebesgue spaces and Vitali convergence theorem. The condition (1.2.14) allows to show that the sequence of approximate nonlinear terms is convergent. The existence result for the case when g depends on both u and $|\nabla u|$ is new, even in the case $\mu=0$. The uniqueness in case the nonlinear term depends only on u is based on Kato type inequalities which are achieved due to the monotonicity. The question of uniqueness has remained open for a while, even when $\mu=0$, in case the gradient $|\nabla u|$ is involved in the analysis due to the lack of monotonicity. In [**105**], we obtain the uniqueness for the case $g(u,|\nabla u|)=u^p+|\nabla u|^q$ and $\mu=0$. One of the main thrusts of our thesis is to obtain the uniqueness in the case $g(u,|\nabla u|)=u^p|\nabla u|^q$ and $0<\mu\le\frac{1}{4}$ in which the interplay between u^p and $|\nabla u|^q$ drastically complicates the situation. Our existence and uniqueness results cover well known results in case $\mu=0$ and provide a full understanding in the case $0\le\mu\le\frac{1}{4}$.

Also in the subcritical range, we provide a complete classification of solutions of (1.2.6) with an isolated boundary singularity. In particular, we show that there are actually two types of solutions with an isolated boundary singularity at a point $y\in\partial\Omega$: the weakly singular solutions, i.e. solutions $u_{k\delta_y}$ of (1.2.8) with plus sign and $\nu=k\delta_y$ with $k>0$ and δ_y being the Dirac

[1] The cases $g(u)=|u|^{p-1}u$ and $g(|\nabla u|)=|\nabla u|^q$ are particular cases of $g(u,|\nabla u|)=u^p|\nabla u|^q$ with $q=0$ and $p=0$ respectively, however we include these cases in the table for the sake of completeness.

mass concentrated at y, and the strongly singular solution $u_{\infty\delta_y}$ which is the limit $u_{\infty\delta_y} = \lim_{k\to\infty} u_{k\delta_y}$ (see Theorem 1.17 and Theorem 1.18).

In the supercritical range, we prove that boundary singularities are *removable*. It means if E is a compact subset of $\partial\Omega$ which has a zero capacity in certain sense and $u \in C(\overline{\Omega} \setminus E)$ is a nonnegative solution of (1.2.6) with plus sign such that u vanishes on $\partial\Omega \setminus E$, then u is identically zero (see Theorem 1.19 and Theorem 1.20).

Source case. Next we consider the source case, i.e. equation (1.2.6) with minus sign. Phenomena occurring in this case are sharply different from those in the absorption case. A striking distinction is that the existence for (1.2.8) holds if the norm of ν is small and does not hold if the norm of ν is large (see Theorem 1.12 and Theorem 1.21). Moreover, in contrast to the absorption case, in the source case, solutions (if exist) are bounded from below by the solution of (1.2.17). The method used to prove the existence in the source case is different from that in the absorption case due to the nature of the nonlinear term. In fact, when $g(u) = u^p$, we use the sup and super solutions method combined with 3-G estimates to show the existence of the minimal solution. This can be done thanks to the monotonicity of the nonlinear term. In a more general case, or in case g depends also on the gradient $|\nabla u|$, this method is invalid (due to the lack of monotonicity), hence we employ the Schauder fixed point theorem to show the existence under the smallness assumption on the boundary data.

The existence result in the typical models are summarized in the following table [2] in which $\|\nu\|_{\mathfrak{M}(\partial\Omega)}$ denotes the norm of the boundary measure datum ν.

Source case	Existence in subcritical range	Non-existence in supercritical range
$g(u) = u^p$	$p < p_\mu$ and $\|\nu\|_{\mathfrak{M}(\partial\Omega)} \leq \rho^*$	$p < p_\mu$, $\|\nu\|_{\mathfrak{M}(\partial\Omega)} > \rho^*$ or $p \geq p_\mu$
$g(\|\nabla u\|) = \|\nabla u\|^q$	$q < q_\mu$ and $\|\nu\|_{\mathfrak{M}(\partial\Omega)}$ small	Not known yet
$g(u, \|\nabla u\|) = u^p + \|\nabla u\|^q$	$p < p_\mu$ and $q < q_\mu$ and $\|\nu\|_{\mathfrak{M}(\partial\Omega)}$ small	Not known yet
$g(u, \|\nabla u\|) = u^p\|\nabla u\|^q$	$(N+\alpha-2)p + (N+\alpha-1)q < N+\alpha$ and $\|\nu\|_{\mathfrak{M}(\partial\Omega)}$ small	Not known yet

Table 2: Source case

From Table 2, we see that, in the source case, non-existence result holds for the model $g(u) = u^p$ if the norm of ν is large enough and has not been known yet in other typical models.

We obtain various necessary and sufficient conditions in terms of sharp estimates on Green kernel and Martin kernel (see Theorem 1.13 for the case $g(u) = u^p$). We also establish criteria expressed in terms of capacities for the

[2]The cases $g(u) = |u|^{p-1}u$ and $g(|\nabla u|) = |\nabla u|^q$ are particular cases of $g(u, |\nabla u|) = u^p|\nabla u|^q$ with $q = 0$ and $p = 0$ respectively, however we include these cases in the table for the sake of completeness.

existence of (1.2.8) (see Theorem 1.14 for the case $g(u) = u^p$ and Theorem 1.22 for the case g depends on u and $|\nabla u|$). Our results for the case where g depends only on u extend those in [**31, 32, 28**] for $\mu = 0$, and the results for the case where g depends on $|\nabla u|$ are new even for $\mu = 0$.

1.3. Detailed statement of main results

In this section, we first discuss important ingredients for the investigation. Then we provide the detailed statement of our recent results on the boundary value problems for linear and nonlinear equations involving operator L_μ which were established in our papers [**106, 119, 78, 79, 80**].

1.3.1. Ingredients. Main ingredients in the study of boundary value problems with measure data for equations (1.2.7) and (1.2.6) include eigen pair of $-L_\mu$ and the Green kernel and the Martin kernel of $-L_\mu$.

First eigenvalue and eigenfunction. The eigenvalue problem associated to $-L_\mu$ is

$$\lambda_\mu := \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla \varphi|^2 - \frac{\mu}{\delta^2}\varphi^2)dx}{\int_\Omega \varphi^2 dx}. \tag{1.3.1}$$

If $\mu < \frac{1}{4}$ then by [**36**, Remark 3.2], problem (1.3.1) admits a positive minimizer φ_μ in $H_0^1(\Omega)$ and hence λ_μ is the first eigenvalue of $-L_\mu$ in $H_0^1(\Omega)$. The function φ_μ, normalized by $\int_\Omega (\varphi_\mu^2/\delta^2)dx = 1$, is the corresponding positive eigenfunction and satisfies

$$-L_\mu \varphi_\mu = \lambda_\mu \varphi_\mu \quad \text{in } \Omega.$$

Moreover, by [**66**] (see also [**108**, Lemmas 5,1, 5.2] and [**50**, Lemma 7] for an alternative proof), $\varphi_\mu \sim \delta^\alpha$ in Ω, where α is given in (1.2.9). It is noted that, since $\mu \in (0, \frac{1}{4}]$, $\alpha \in [\frac{1}{2}, 1)$.

If $\mu = \frac{1}{4}$ then there is no minimizer of (1.3.1) in $H_0^1(\Omega)$, but there exists a nonnegative function $\varphi_{\frac{1}{4}} \in H_{loc}^1(\Omega)$ such that $-L_{\frac{1}{4}}\varphi_{\frac{1}{4}} = \lambda_{\frac{1}{4}}\varphi_{\frac{1}{4}}$ in Ω in the sense of distributions. Again by [**66**], $\varphi_{\frac{1}{4}} \sim \delta^{\frac{1}{2}}$ in Ω.

We observe from (1.2.3) and (1.3.1) that $\lambda_\mu > 0$ if $\mu < C_H(\Omega)$, $\lambda_\mu = 0$ if $\mu = C_H(\Omega) < \frac{1}{4}$, while $\lambda_\mu < 0$ when $\mu > C_H(\Omega)$. It is not known if $\lambda_\mu > 0$ when $\mu = C_H(\Omega) = \frac{1}{4}$. However, if Ω is convex or if $-\Delta\delta \geq 0$ in the sense of distributions – in these cases $C_H(\Omega) = \frac{1}{4}$ – then $\lambda_{\frac{1}{4}} > 0$ (see [**36**, Thereom II]) and [**16**, Theorem A with $k = 1$ and $p = 2$]).

Green kernel and Martin kernel. The positivity of the first eigenvalue λ_μ plays a crucial role in derivation of the existence of Green kernel and Martin kernel. From the above observation, we see that this property does not hold for arbitrary $\mu \in (0, \frac{1}{4}]$. Therefore, in order to go further in the study of the Green kernel and Martin kernel, we assume that

$$\mu \in (0, \frac{1}{4}] \quad \text{and} \quad \lambda_\mu > 0. \tag{1.3.2}$$

Notice that this assumption (1.3.2) is fulfilled when $\mu \in (0, C_H(\Omega))$. *Throughout the thesis, unless otherwise stated, we assume that* (1.3.2) *holds.*

We say that u is L_μ-harmonic (resp. L_μ-subharmonic, L_μ-superharmonic) in Ω if $u \in L^1_{loc}(\Omega)$ and

$$-L_\mu u = 0 \text{ (resp. } -L_\mu u \le 0, -L_\mu u \ge 0)$$

in the sense of distributions in Ω, namely

$$-\int_\Omega u L_\mu \varphi \, dx = (\text{resp.} \le, \ge)\, 0, \quad \forall \varphi \in C_c^\infty(\Omega).$$

Under assumption (1.3.2), φ_μ is a positive L_μ-superharmonic function in Ω. Therefore, a classical result of Ancona [8] and a result of Gkikas and Véron [77, Section 2] (for the case $\mu = \frac{1}{4}$) imply that for every $y \in \partial\Omega$, there exists a positive L_μ-harmonic function in Ω which vanishes on $\partial\Omega \setminus \{y\}$ and is unique up to a constant. This function is denoted by $K_\mu^\Omega(\cdot, y)$ with normalization $K_\mu^\Omega(x_0, y) = 1$ where $x_0 \in \Omega$ is a fixed reference point. The function $(x, y) \mapsto K_\mu^\Omega(x, y)$, $(x, y) \in \Omega \times \partial\Omega$, is called the L_μ-*Martin kernel* in Ω relative to x_0. We emphasize that the role of the Martin kernel is similar to that of the Poisson kernel in case $\mu = 0$. However, unlike the Poisson kernel which has a finite mass, the Martin kernel $K_\mu^\Omega(, y)$ may have zero or infinite mass at y. In particular, if $\mu \in (0, C_H(\Omega))$, then the mass of $K_\mu^\Omega(\cdot, y)$ at y is zero and hence the Poisson kernel does not exist in this case.

Furthermore, by [8] and [77, Theorem 2.33] (for $\mu = \frac{1}{4}$), there is a one-to-one correspondence between the set of positive L_μ-harmonic functions and the set of positive bounded Radon measures on $\partial\Omega$. More precisely, we have:

THEOREM 1.2 (Representation Theorem). *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ the function*

$$\mathbb{K}_\mu^\Omega[\nu](x) := \int_{\partial\Omega} K_\mu^\Omega(x, y) d\nu(y) \quad \forall x \in \Omega \tag{1.3.3}$$

is L_μ-harmonic in Ω. Conversely, for every positive L_μ-harmonic function u in Ω there exists a unique measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $u = \mathbb{K}_\mu^\Omega[\nu]$ in Ω.

The measure ν such that $u = \mathbb{K}_\mu^\Omega[\nu]$ is called the L_μ-*boundary measure of* u. If $\mu = 0$, ν becomes the m-boundary trace of u (see the definition of m-boundary trace in Definition 1.1). However, when $\mu \in (0, C_H(\Omega))$, it can be proved that, for every $\nu \in \mathfrak{M}^+(\partial\Omega)$, the m-boundary trace of $\mathbb{K}_\mu^\Omega[\nu]$ is zero.

Let G_μ^Ω be the Green kernel for the operator $-L_\mu$ in $\Omega \times \Omega$ defined by

$$G_\mu^\Omega(x, y) = \int_0^\infty H_\mu(x, y, t) dt$$

where H_μ is the heat kernel associated with $-L_\mu$. By [66, Theorem 4.11], for every $x, y \in \Omega$, $x \ne y$,

$$G_\mu^\Omega(x, y) \sim |x - y|^{2-N} \left(1 \wedge \delta(x)^\alpha \delta(y)^\alpha |x - y|^{-2\alpha}\right). \tag{1.3.4}$$

This estimate leads to the observation that a measure $\tau \in \mathfrak{M}^+(\Omega, \delta^\alpha)$ if and only if $\mathbb{G}_\mu^\Omega[\tau]$ is finite a.e. in Ω (see [106, page 9]) where $\mathbb{G}_\mu^\Omega$ is the Green operator acting on τ defined by

$$\mathbb{G}_\mu^\Omega[\tau](x) := \int_\Omega G_\mu^\Omega(x,y)d\tau(y).$$

From the following relation between Green kernel and Martin kernel

$$K_\mu^\Omega(x,y) = \lim_{z \to y} \frac{G_\mu^\Omega(x,z)}{G_\mu^\Omega(x_0,z)} \quad \forall x \in \Omega, y \in \partial\Omega,$$

and estimate (1.3.4), we deduce

$$K_\mu^\Omega(x,y) \sim \delta(x)^\alpha |x-y|^{2-N-2\alpha} \quad \forall x \in \Omega, y \in \partial\Omega. \tag{1.3.5}$$

Estimates in weak Lebesgue spaces. Let us first recall the definition of weak Lebesgue spaces (or Marcinkiewicz spaces). See the paper of Bénilan, Brezis and Crandall [21] for more details of these spaces.

Let $\kappa > 1$, $\kappa' = \frac{\kappa}{\kappa-1}$ and ϕ be a positive weight function. We set

$$L_w^\kappa(\Omega,\phi) := \left\{ u \in L^1_{loc}(\Omega) : \|u\|_{L_w^\kappa(\Omega,\phi)} < \infty \right\}$$

where

$$\|u\|_{L_w^\kappa(\Omega,\phi)} := \inf \left\{ c \in [0,\infty] : \int_E |u|\phi dx \le c \left(\int_E \phi dx \right)^{\frac{1}{\kappa'}}, \forall \text{ Borel } E \subset \Omega \right\}.$$

The space $L_w^\kappa(\Omega,\phi)$ is called the weak Lebesgue space with exponent κ (or Marcinkiewicz space) with the norm $\|.\|_{L_w^\kappa(\Omega,\phi)}$. The subscript w in the notation stands for 'weak'. The relation between the Lebesgue space norm and the weak Lebesgue space norm is given in [21, Lemma A.2(ii)]. In particular, for any $1 \le r < \kappa < \infty$ and $u \in L^1_{loc}(\Omega)$, there exists $C(r,\kappa) > 0$ such that for any Borel subset E of Ω

$$\int_E |u|^r \phi dx \le C(r,\kappa) \|u\|^r_{L_w^\kappa(\Omega,\phi)} \left(\int_E \phi dx \right)^{1-\frac{r}{\kappa}}.$$

We notice that $L^\kappa(\Omega,\phi) \subset L_w^\kappa(\Omega,\phi) \subset L^r(\Omega,\phi)$ for all $1 \le r < \kappa$. Weak Lebesgue spaces play an important role because they provide optimal estimates in the study of nonlinear elliptic equations in a measure framework.

Sharp estimates on Green kernel and Martin kernel in weak Lebesgue spaces were obtained in [106, Proposition 2.8] and [78, Proposition 2.4]. By using estimates (1.3.4) and (1.3.5), together with a key lemma in Bidaut-Véron and Vivier [31, Lemma 2.4], we show that (see [78, Proposition 2.4]) there exists a constant $c = c(N,\mu,\Omega)$ such that

$$\|\mathbb{G}_\mu^\Omega[\tau]\|_{L_w^{\frac{N+\alpha}{N+\alpha-2}}(\Omega,\delta^\alpha)} \le c\|\tau\|_{\mathfrak{M}(\Omega,\delta^\alpha)} \quad \forall \tau \in \mathfrak{M}(\Omega,\delta^\alpha), \tag{1.3.6}$$

$$\|\mathbb{K}_\mu^\Omega[\nu]\|_{L_w^{\frac{N+\alpha}{N+\alpha-2}}(\Omega,\delta^\alpha)} \le c\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \forall \nu \in \mathfrak{M}(\partial\Omega). \tag{1.3.7}$$

The above estimates indicate that the value $\frac{N+\alpha}{N+\alpha-2}$ would be a critical exponent for semilinear equations with the nonlinear term depending on solutions.

When the nonlinear term depends on the gradient of solutions, the estimates on the gradient of the Green kernel and Martin kernel also play an important role in the analysis. More precisely, we prove that there exists a positive constant $c = c(N, \mu, \Omega)$ such that

$$\|\nabla \mathbb{G}_\mu^\Omega[|\tau|]\|_{L_w^{\frac{N+\alpha}{N+\alpha-1}}(\Omega,\delta^\gamma)} \le c\|\tau\|_{\mathfrak{M}(\Omega,\delta^\alpha)} \quad \forall \tau \in \mathfrak{M}(\Omega, \delta^\alpha), \tag{1.3.8}$$

$$\|\nabla \mathbb{K}_\mu^\Omega[|\nu|]\|_{L_w^{\frac{N+\alpha}{N+\alpha-1}}(\Omega,\delta^\alpha)} \le c\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \forall \nu \in \mathfrak{M}(\partial\Omega), \tag{1.3.9}$$

where

$$\nabla \mathbb{G}_\mu^\Omega[\tau](x) = \int_\Omega \nabla_x G_\mu^\Omega(x,y)d\tau(y)$$
$$\nabla \mathbb{K}_\mu[\nu](x) = \int_{\partial\Omega} \nabla_x K_\mu^\Omega(x,z)d\nu(z).$$

Estimates (1.3.8) and (1.3.9) indicate that the value $\frac{N+\alpha}{N+\alpha-1}$ would be a critical exponent for semilinear equations with the nonlinear term depending on the gradient of solutions.

1.3.2. Notions of μ-boundary trace. One of the first attempt in the study of boundary value problems for linear and nonlinear equations involving operator L_μ was carried out by Bandle, Moroz and Reichel [11] who investigated L_μ-sub and superharmonic functions and obtained the existence and nonexistence of large solutions, i.e. solutions blowing up on the boundary $\partial\Omega$. Further research related to large solutions is due to Bandle and Pozio [12, 13] and Du and Wei [52]. However, there are other types of solutions which may not singular on the whole boundary $\partial\Omega$. Therefore, in order to characterize the boundary behavior of such solutions to equations with a Hardy potential, we need a new tool.

Normalized boundary trace. An interesting observation emerging from estimate (1.3.5) and (1.3.3) is that there exists $\beta_0 > 0$ small enough such that for any $\nu \in \mathfrak{M}^+(\partial\Omega)$ and any $\beta \in (0, \beta_0)$, there holds

$$\int_{\Sigma_\beta} \mathbb{K}_\mu^\Omega[\nu]dS \sim \beta^{1-\alpha}\|\nu\|_{\mathfrak{M}(\partial\Omega)},$$

where dS denotes the surface element on $\Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\}$. This, together with the fact $\alpha < 1$, implies that the m-boundary trace (see Definition 1.1) of $\mathbb{K}_\mu^\Omega[\nu]$ is zero for any $\nu \in \mathfrak{M}^+(\partial\Omega)$. Therefore, the notion of m-boundary trace (see Definition 1.1) is no longer an appropriate tool to describe the boundary behavior of $\mathbb{K}_\mu^\Omega[\nu]$. Taking into account of the Representation theorem (see Theorem 1.2), we see that this notion does not play a role in the study of L_μ-harmonic functions. Therefore, we introduce a new notion (see [106, Definition 1.2]) as follows:

Definition 1.3. Assume $0 < \mu < C_H(\Omega)$. A function $u \in W^{1,r}_{loc}(\Omega)$ with some $r > 1$ possesses a *normalized μ-boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that

$$\lim_{\beta\to 0} \beta^{\alpha-1} \int_{\Sigma_\beta} |u - \mathbb{K}_\mu^\Omega[\nu]|dS = 0, \tag{1.3.10}$$

where dS denotes the surface element on Σ_β. The normalized μ-boundary trace will be denoted by $\mathrm{tr}_\mu^*(u)$.

The terminology 'normalized' comes from the term $\beta^{\alpha-1}$ in (1.3.10). Roughly speaking, the difference between the function u and $\mathbb{K}_\mu[\nu]$ on Σ_β has to be normalized by the weight $\beta^{\alpha-1} = \delta(x)^{\alpha-1}$ for $x \in \Sigma_\beta$ so that it tends to zero. The weight $\beta^{\alpha-1}$ is also 'propositional' to the volume of the surface Σ_β. The term on the left-hand side of (1.3.10) can be understood as the 'average' of the difference between u and $\mathbb{K}_\mu[\nu]$ near the boundary.

This notion is well-defined, i.e. if ν and ν' satisfy (1.3.10) then $\nu = \nu'$ (see the remark after [**106**, Definition 1.2]). The condition $u \in W^{1,r}_{loc}(\Omega)$ ensures the meaning of u on Σ_β as the Sobolev trace of u on Σ_β. The restriction $0 < \mu < C_H(\Omega)$ in [**106**] is imposed to guarantee that the first eigenfunction φ_μ of $-L_\mu$ is a positive L_μ-superharmonic in Ω. The notion was extended to the range $\mu < \frac{1}{4}$ by Marcus and Moroz in [**104**] due the fact that there exist local L_μ-superharmonic functions in a neighborhood of $\partial\Omega$. The notion is new even in the case $\mu = 0$ and enables us to measure how close a function is in comparison with the Martin kernel near the boundary.

An important feature of this notion is that it allows to derive $\mathrm{tr}^*_\mu(\mathbb{G}^\Omega_\mu[\tau]) = 0$ for any $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$ and $\mathrm{tr}^*_\mu(\mathbb{K}^\Omega_\mu[\nu]) = \nu$ for any $\nu \in \mathfrak{M}(\partial\Omega)$.

Boundary trace defined in a dynamic way. In parallel, Gkikas and Véron [**77**] introduced another notion of boundary trace which is defined using the weak convergence of measures. Let $D \Subset \Omega$ and $x_0 \in D$ be a fixed reference point. If $h \in C(\partial D)$ then the following problem

$$\begin{cases} -L_\mu u = 0 & \text{in } D, \\ u = h & \text{on } \partial D, \end{cases} \tag{1.3.11}$$

admits a unique solution which allows to define the L_μ*-harmonic measure* $\omega^{x_0}_D$ on ∂D by

$$u(x_0) = \int_{\partial D} h(y) d\omega^{x_0}_D(y). \tag{1.3.12}$$

Let $\{\Omega_n\}$ be a C^2 *exhaustion* of Ω. For each n, let $\omega^{x_0}_{\Omega_n}$ be the $L^{\Omega_n}_\mu$-harmonic measure on $\partial\Omega_n$.

Definition 1.4. Let $\mu \in (0, \frac{1}{4}]$. We say that a function u possesses a μ*-boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that for any C^2 exhaustion $\{\Omega_n\}$ of Ω, there holds

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \phi u \, d\omega^{x_0}_{\Omega_n} = \int_{\partial\Omega} \phi \, d\nu \quad \forall \phi \in C(\overline{\Omega}). \tag{1.3.13}$$

The μ-boundary trace of u is denoted by $\mathrm{tr}_\mu(u)$ and we write $\mathrm{tr}_\mu(u) = \nu$.

The advantage of this notion is that it does not require to determine the normalization factor in the definition, however it does not provide information on the boundary behavior of functions near $\partial\Omega$.

Equivalence of the notions of μ-boundary trace. We show (see [**78**]) that the normalized μ-boundary trace in Definition 1.3 and the μ-boundary trace in Definition 1.4 are actually equivalent. This is achieved thanks to the following result (see Section 2, especially Proposition 2.5 and Proposition

2.6 in [77])

$$\begin{cases} \mathrm{tr}^*_\mu(\mathbb{G}^\Omega_\mu[\tau]) = \mathrm{tr}_\mu(\mathbb{G}^\Omega_\mu[\tau]) = 0 & \forall \tau \in \mathfrak{M}(\Omega, \delta^\alpha), \\ \mathrm{tr}^*_\mu(\mathbb{K}^\Omega_\mu[\nu]) = \mathrm{tr}_\mu(\mathbb{K}^\Omega_\mu[\nu]) = \nu & \forall \nu \in \mathfrak{M}(\partial\Omega). \end{cases}$$

Thanks to this result, these notions can be used interchangeably. In the sequel, we employ the notion of μ-boundary trace given in Definition 1.4 in the study of linear equations and nonlinear equations.

1.3.3. Linear equations. In this subsection, we consider nonhomogeneous linear equations of the form

$$-L_\mu u = \tau \quad \text{in } \Omega \tag{1.3.14}$$

with $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$. The boundary value problem for (1.3.14) with μ-boundary trace is formulated as

$$\begin{cases} -L_\mu u = \tau & \text{in } \Omega \\ \mathrm{tr}_\mu(u) = \nu, \end{cases} \tag{1.3.15}$$

where $\nu \in \mathfrak{M}(\partial\Omega)$.

Definition 1.5. Assume $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$ and $\nu \in \mathfrak{M}(\partial\Omega)$.

(i) A function u is a solution of (1.3.14) if $u \in L^1_{loc}(\Omega)$ and u satisfies (1.3.14) in the sense of distributions in Ω, namely

$$-\int_\Omega u L_\mu \phi dx = \int_\Omega \phi d\tau \quad \forall \phi \in C^\infty_c(\Omega).$$

(ii) A function u is a solution of (1.3.15) if u is a solution of (1.3.14) and $\mathrm{tr}_\mu(u) = \nu$.

Our main results regarding this problem provides a full understanding of equation (1.3.14) and problem (1.3.15), as listed below.

Theorem 1.6. *(i) For any $\nu \in \mathfrak{M}(\partial\Omega)$, the function $u = \mathbb{K}^\Omega_\mu[\nu]$ is the unique solution of problem (1.3.15) with $\tau = 0$. If u is a nonnegative L_μ-harmonic function and $\mathrm{tr}_\mu(u) = 0$ then $u = 0$.*

(ii) For any $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$, the function $u = \mathbb{G}^\Omega_\mu[\tau]$ is the unique solution of (1.3.15) with $\nu = 0$. In particular, $\mathrm{tr}_\mu(\mathbb{G}^\Omega_\mu[\tau]) = 0$.

(iii) Let u be a positive L_μ-subharmonic function. If u is dominated by an L_μ-superharmonic function then $L_\mu u \in \mathfrak{M}^+(\Omega, \delta^\alpha)$ and u has a μ-boundary trace. In this case $\mathrm{tr}_\mu(u) = 0$ if and only if $u \equiv 0$.

(iv) Let u be a positive L_μ-superharmonic function. Then there exist $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$ such that

$$u = \mathbb{G}^\Omega_\mu[\tau] + \mathbb{K}^\Omega_\mu[\nu]. \tag{1.3.16}$$

In particular, u is an L_μ-potential (i.e., u does not dominate any positive L_μ-harmonic function) if and only if $\mathrm{tr}_\mu(u) = 0$.

(v) For every $\nu \in \mathfrak{M}(\partial\Omega)$ and $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$, problem (1.3.15) admits a unique solution. The solution is given by (1.3.16).

(vi) u is a solution of of (1.3.15) if and only if $u \in L^1(\Omega, \delta^\alpha)$ and

$$-\int_\Omega u L_\mu \phi dx = \int_\Omega \phi d\tau - \int_\Omega \mathbb{K}^\Omega_\mu[\nu] L_\mu \phi dx \quad \forall \phi \in X(\Omega). \tag{1.3.17}$$

where $X_\mu(\Omega)$ *is the space of admissible test functions defined by*

$$X_\mu(\Omega) := \{\phi \in H^1_{loc}(\Omega) : \delta^{-\alpha}\phi \in H^1(\Omega, \delta^{2\alpha}),\ L_\mu\phi \in L^\infty(\Omega, \delta^{-\alpha})\}. \tag{1.3.18}$$

Let us comment on the formula (1.3.17). Since $u \in L^1(\Omega, \delta^\alpha)$ and $L_\mu\phi \in L^\infty(\Omega, \delta^{-\alpha})$, it follows that $uL_\mu\phi \in L^1(\Omega)$, hence the term on the left-hand side of (1.3.17) is finite. Next, since $\phi \in X_\mu(\Omega)$, by [**77**, Lemma 3.1], we have $|\phi| \le c\delta^\alpha$, and hence the first term on the right-hand side of (1.3.17) is finite. Finally, since $\mathbb{K}^\Omega_\mu[\nu] \in L^1(\Omega, \delta^\alpha)$ (see [**78**, Proposition 2.4]) and $L_\mu\phi \in L^\infty(\Omega, \delta^{-\alpha})$, the the second term on the right-hand side of (1.3.17) is finite.

Theorem 1.6 was obtained in our joint work with Marcus [**105**, Proposition I.] in case $\mu \in (0, C_H(\Omega))$, dealing with the notion of normalized μ-boundary trace and the space of test functions

$$Y_\mu(\Omega) := \{\zeta \in C^2(\Omega) : L_\mu\zeta \in L^\infty(\Omega, \delta^{1-\alpha}),\ \zeta \in L^\infty(\Omega, \delta^{-\alpha})\}$$

instead of $X_\mu(\Omega)$, and then was extended in our joint work with Gkikas [**78**, Section 2] to the case $\mu \in (0, \frac{1}{4}]$, dealing with the notion of μ-boundary trace. The above results generalize those for Laplacian in measure frameworks (see [**116**]).

The theory for linear equation (1.3.14) forms a basis for the study of nonlinear equation (1.2.6). The boundary value problem for (1.2.6) with a given μ-boundary trace is formulated as

$$\begin{cases} -L_\mu u \pm g(u, |\nabla u|) = 0 & \text{in } \Omega, \\ \operatorname{tr}_\mu(u) = \nu. \end{cases} \tag{1.3.19}$$

Let us give the definition of weak solutions of equation (1.2.6) and problem (1.3.19).

Definition 1.7. (i) A function u is a weak solution of (1.2.6) if $u \in L^1_{loc}(\Omega)$, $g(u, |\nabla u|) \in L^1_{loc}(\Omega)$ and u satisfies (1.2.6) in the sense of distributions in Ω, namely

$$-\int_\Omega uL_\mu\phi dx \pm \int_\Omega g(u, |\nabla u|)\phi dx = 0 \quad \forall \phi \in C^\infty_c(\Omega).$$

(ii) Let $\nu \in \mathfrak{M}(\partial\Omega)$. A function u is a weak solution to of (1.3.19) if u is a weak solution of (1.2.6) and $\operatorname{tr}_\mu(u) = \nu$.

In the spirit of Theorem 1.6, it is interesting to ask if every weak solution of (1.3.19) satisfies the decomposition

$$u \pm \mathbb{G}^\Omega_\mu[g(u, |\nabla u|)] = \mathbb{K}^\Omega_\mu[\nu] \quad \text{in } \Omega \tag{1.3.20}$$

and the weak formula

$$-\int_\Omega uL_\mu\phi dx \pm \int_\Omega g(u, |\nabla u|)\phi)dx = -\int_\Omega \mathbb{K}^\Omega_\mu[\nu]L_\mu\phi dx \quad \forall \phi \in X_\mu(\Omega). \tag{1.3.21}$$

The answer to this question is positive. We will discuss it the typical models successively.

1.3.4. Absorption term. In case $g(u, |\nabla u|) = |u|^{p-1}u$, with $p > 1$, and the plus sign occurs in (1.2.6), equation (1.2.6) and problem (1.3.19) become

$$- L_\mu u + |u|^{p-1}u = 0 \quad \text{in } \Omega \tag{1.3.22}$$

and

$$\begin{cases} -L_\mu u + |u|^{p-1}u = 0 & \text{in } \Omega \\ \operatorname{tr}_\mu(u) = \nu, \end{cases} \tag{1.3.23}$$

where $\nu \in \mathfrak{M}(\partial\Omega)$.

As in the the case $\mu = 0$, the first step to analyze the μ-boundary trace of solutions to (1.3.22) is to deal with moderate solutions of (1.3.22).

Definition 1.8. A function u is a *moderate solution* of (1.3.22) if $|u| \leq v$ where v is a positive L_μ-harmonic function.

Our next theorem describes the relation between the concept of μ-boundary trace and the notion of moderate solutions of (1.3.22).

Theorem 1.9. *Assume $p > 1$ and let u be a positive solution of* (1.3.22). *Then the following statements are equivalent.*

(i) u is a L_μ-moderate solution of (1.3.22).

(ii) u admits a μ-boundary trace $\nu \in \mathfrak{M}^+(\partial\Omega)$. It means u is a solution of (1.3.23).

(iii) $u \in L^p(\Omega, \delta^\alpha)$ and (1.3.20) *holds with $g(u, |\nabla u|) = u^p$ and the plus sign, where $\nu = \operatorname{tr}_\mu(u)$.*

Furthermore, a positive function u is a solution of (1.3.23) *if and only if $u \in L^p(\Omega, \delta^\alpha)$ and* (1.3.21) *holds with $g(u, |\nabla u|) = u^p$ and the plus sign.*

This theorem is a combination of [**106**, Proof of Theorem A] and [**77**, Proof of Proposition 3.5] and covers the previous results for the case $\mu = 0$ in [**116**, Section 2.1]. The proof is based on Theorem 1.6, Representation Theorem 1.2 and an approximation process.

A remarkable distinction in the study of nonlinear problem (1.3.23) in comparison with that of linear problem (1.3.15) is that problem (1.3.23) is solvable for any $\nu \in \mathfrak{M}(\partial\Omega)$ only if the nonlinear term does not grow 'too fast'. More precisely, we show that the exponent p_μ given in (1.2.11) is a critical exponent for the existence of solutions to (1.3.23) in the sense that if $1 < p < p_\mu$ then problem (1.3.23) has a unique solution for every measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ while, if $p \geq p_\mu$ then the problem has no solution if ν is a Dirac measure. This is reflected in the following theorem.

Theorem 1.10. *Assume $p > 1$.*

(i) *Existence, uniqueness and stability. If $1 < p < p_\mu$ then for every $\nu \in \mathfrak{M}(\partial\Omega)$* (1.3.23) *admits a unique positive solution.*

(i) *Non-existence. If $p \geq p_\mu$ then for every $k > 0$ and $y \in \partial\Omega$, there is no positive solution of* (1.3.23) *with μ-boundary trace $k\delta_y$, where δ_y denotes the Dirac measure concentrated at y.*

It is noticed that for any $\nu \in L^1(\partial\Omega)$, problem (1.3.23) admits a unique solution.

We proved Theorem 1.10 for $\mu \in (0, C_H(\Omega))$ in connection with normalized μ-boundary trace (see [**106**, Theorem E and Theorem F]). The results were extended by Marcus and Moroz [104] to the case $\mu < \frac{1}{4}$ for arbitrary domains, without any requirement on the positivity of the first eigenvalue λ_μ due to the key observation that there exists a positive *local* L_μ*-superharmonic function* in the whole range $\mu < \frac{1}{4}$.

In parallel, Gkikas and Véron considered the semilinear equations with a class of more general absorption terms $g(u)$ where $g : \mathbb{R} \to \mathbb{R}$ is a continuous, nondecreasing function with $g(0) = 0$, under assumption (1.3.2). They pointed out in [77, Theorem 3.3] that, under the condition

$$\int_1^\infty (g(t) + |g(-t)|)t^{-1-p_\mu} dt < \infty, \tag{1.3.24}$$

for every $\nu \in \mathfrak{M}(\partial\Omega)$ there exists a unique weak solution of (1.3.19) with nonlinear term $g(u)$. Clearly, when $g(u) = |u|^{p-1}u$, with $p > 1$, condition (1.3.24) is translated as $1 < p < p_\mu$ and the existence result by Gkikas and Véron covers statement (i) of Theorem 1.10. In particular, the result asserts that for any $k > 0$, there exists a unique solution $u_{k\delta_y}$ of (1.3.23) with $\nu = k\delta_y$. It was also showed that the function $u_{\infty\delta_y} := \lim_{k\to\infty} u_{k\delta_y}$ is a solution to the equation in (1.3.22) which vanishes on $\partial\Omega \setminus \{y\}$ and admits a strong singularity at y. When $p \geq p_\mu$, they provided a necessary and sufficient condition expressed in terms of Besov capacities for the existence of a solution to (1.3.23), which includes statement (ii) of Theorem 1.10 as a concrete case.

1.3.5. Source term. We are also interested in semilinear elliptic equations with a source term

$$-L_\mu u = g(u) \quad \text{in } \Omega \tag{1.3.25}$$

and the associated boundary value problem

$$\begin{cases} -L_\mu u = g(u) & \text{in } \Omega \\ \mathrm{tr}_\mu(u) = \nu \end{cases} \tag{1.3.26}$$

where $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function with $g(0) = 0$. When dealing with (1.3.25) and (1.3.26), one encounters the following difficulties. The first one stems from the lack of a universal upper bound for solutions of (1.3.25). The second difficulty is that $\mathbb{K}_\mu^\Omega[\nu]$ is a subsolution of (1.3.26) and therefore it is no longer an upper bound, but a lower bound for solutions of (1.3.26).

We also show that weak solutions of (1.3.26) can be represented as in (1.3.20), namely they can be decomposed as the sum of two terms: the action of Green operator on the nonlinearity and the action of the Martin kernel on the boundary datum. Equivalently, it also means that weak solutions satisfy an integral formulation weakfor-ugradu (See [119, Theorem A] for $\mu \in (0, C_H(\Omega))$ and [78, Proposition A] for $\mu \in (0, \frac{1}{4}]$ and more general results involving also interior measure data).

A counterpart of Theorem 1.9 is also obtained for (1.3.25) and (1.3.26) (see [77, Proposition A]). Based on this, together with weak Lebesgue space estimate on Green kernel and Martin kernel and the Schauder fixed point theorem, we show that the value p_μ given in (1.2.11) is also a critical exponent for the existence of (1.3.26), as stated below (see [**119**, Theorem H and Theorem I]).

THEOREM 1.11. (i) *Subcritical case. Assume that*

$$\Lambda_0 := \int_1^\infty g(t)t^{-1-p_\mu}dt < \infty, \tag{1.3.27}$$

and $0 \le g(t) \le \Lambda_1 t^{p_1} + \theta$ *for all* $t \in [0,1]$ *with some* $p_1 > 1, \Lambda_1 > 0, \theta > 0$. *There exist* $\theta_0 > 0$ *and* $\rho_0 > 0$ *such that for any* $\theta \in (0,\theta_0)$ *and* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *such that* $\|\nu\|_{\mathfrak{M}(\partial\Omega)} < \rho_0$, *problem* (1.3.26) *admits a nonnegative weak solution.*

(ii) *Sublinear case. Assume that*

$$0 \le g(t) \le \Lambda_2 t^{p_2} + \theta \quad \forall t \ge 0 \tag{1.3.28}$$

for some $p_2 \in (0,1]$, $\Lambda_2 > 0$ *and* $\theta > 0$. *In* (1.3.28), *if* $p_2 = 1$ *we assume in addition that* Λ_2 *is small enough. Then for any* $\nu \in \mathfrak{M}^+(\partial\Omega)$, (1.3.26) *admits a nonnegative weak solution.*

When $g(u) = u^p$, we obtain a deeper analysis of the existence and nonexistence phenomena. Indeed, we prove (see [**119**, Theorem D and Theorem G]) the existence of a threshold value for the existence of solutions to

$$\begin{cases} -L_\mu u = u^p & \text{in } \Omega, \\ \mathrm{tr}_\mu(u) = \rho\nu, \end{cases} \tag{1.3.29}$$

where $\rho > 0$ is a parameter and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$.

THEOREM 1.12. *Let* $p > 1$ *and* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *with* $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$.

I. *Subcritical case:* $p \in (1, p_\mu)$. *There exists* $\rho^* \in (0,\infty)$ *such that the followings hold.*

(i) *If* $\rho \in (0,\rho^*]$ *then problem* (1.3.29) *admits a minimal positive weak solution* $\underline{u}_{\rho\nu}$.
(ii) *If* $\rho > \rho^*$ *then* (1.3.29) *does not admit any positive solution.*

II. *Supercritical case:* $p \ge p_\mu$. *For every* $\rho > 0$ *and* $y \in \partial\Omega$, *there is no positive weak solution of* (1.3.29) *with* $\nu = \delta_y$, *where* δ_y *is the Dirac mass concentrated at* $y \in \partial\Omega$.

Theorem 1.12 shows a sharp difference between the absorption case and the source case. More clearly, in the source case, the existence depends not only on the concentration of the boundary datum but also on its norm. It was proved later on in our recent paper [**23**] that the multiplicity occurs when $\rho \in (0,\rho^*)$ and the uniqueness happens when $\rho = \rho^*$.

We also established various necessary and sufficient conditions in terms of estimates on Green kernel (see [**78**, Theorem B]) for the existence for the Dirichlet problem with interior measure data and boundary measure data

$$\begin{cases} -L_\mu u = u^p + \sigma\tau & \text{in } \Omega, \\ \mathrm{tr}_\mu(u) = \rho\nu. \end{cases} \tag{1.3.30}$$

Existence results are stated in the following theorem.

THEOREM 1.13. *Let $\tau \in \mathfrak{M}^+(\Omega, \delta^\alpha)$, $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $p > 0$.*

(i) Assume $0 < p < p_\mu$. Then there exists a constant $C > 0$ such that

$$\mathbb{G}_\mu^\Omega[\mathbb{K}_\mu^\Omega[\nu]^p] \le C\mathbb{K}_\mu^\Omega[\nu] \quad \text{a.e. in } \Omega, \tag{1.3.31}$$

$$\mathbb{G}_\mu^\Omega[\mathbb{G}_\mu^\Omega[\tau]^p] \le C\mathbb{G}_\mu^\Omega[\tau] \quad \text{a.e. in } \Omega. \tag{1.3.32}$$

(ii) If (1.3.31) and (1.3.32) hold then problem (1.3.30) admits a weak solution u satisfying

$$\mathbb{G}_\mu^\Omega[\sigma\tau] + \mathbb{K}_\mu^\Omega[\rho\nu] \le u \le C(\mathbb{G}_\mu^\Omega[\sigma\tau] + \mathbb{K}_\mu^\Omega[\rho\nu]) \text{ a.e. in } \Omega \tag{1.3.33}$$

for $\sigma > 0$ and $\rho > 0$ small enough if $p > 1$, for any $\sigma > 0$ and $\rho > 0$ if $0 < p < 1$.

(iv) If $p > 1$ and (1.3.30) admits a weak solution then (1.3.31) and (1.3.32) hold with constant $C = \frac{1}{p-1}$.

(v) Assume $0 < p < p_\mu$. Then there exists a constant $C > 0$ such that for any weak solution u of (1.3.30) there holds

$$\mathbb{G}_\mu^\Omega[\sigma\tau] + \mathbb{K}_\mu^\Omega[\rho\nu] \le u \le C(\mathbb{G}_\mu^\Omega[\sigma\tau] + \mathbb{K}_\mu^\Omega[\rho\nu] + \delta^\alpha) \text{ a.e. in } \Omega. \tag{1.3.34}$$

In order to deal with (1.3.30) in the supercritical case, i.e. $p \ge p_\mu$, we make use of interior capacities and boundary capacities which are recalled below. For $0 \le \theta \le \beta < N$, set

$$N_{\theta,\beta}(x,y) := \frac{1}{|x-y|^{N-\beta}\max\{|x-y|, \delta(x), \delta(y)\}^\theta}, \quad \forall (x,y) \in \overline{\Omega} \times \overline{\Omega}, x \ne y, \tag{1.3.35}$$

$$\mathbb{N}_{\theta,\beta}[\tau](x) := \int_{\overline{\Omega}} N_{\theta,\beta}(x,y)d\tau(y), \quad \forall \tau \in \mathfrak{M}^+(\overline{\Omega}). \tag{1.3.36}$$

For $a > -1$, $0 \le \theta \le \beta < N$ and $s > 1$, define $\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}$ by

$$\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E) := \inf\left\{\int_{\overline{\Omega}} \delta^a \phi^s \, dx : \ \phi \ge 0, \ \mathbb{N}_{\theta,\beta}[\delta^a\phi] \ge \chi_E\right\}, \tag{1.3.37}$$

for any Borel set $E \subset \overline{\Omega}$. Here χ_E denotes the indicator function of E.

Next we recall the capacity $\mathrm{Cap}^{\partial\Omega}_{\theta,s}$ introduced in [28] which is used to deal with boundary measures. Let $\theta \in (0, N-1)$ and denote by $\mathcal{B}_\theta$ the Bessel kernel in $\mathbb{R}^{N-1}$ with order θ. For $s > 1$, define

$$\mathrm{Cap}_{\mathcal{B}_\theta,s}(F) := \inf\left\{\int_{\mathbb{R}^{N-1}} \delta^s dy : \ \phi \ge 0, \ \mathcal{B}_\theta * \phi \ge \chi_F\right\} \tag{1.3.38}$$

for any Borel set $F \subset \mathbb{R}^{N-1}$. Since Ω is a bounded smooth domain in $\mathbb{R}^N$, there exist open sets $O_1, ..., O_m$ in $\mathbb{R}^N$, diffeomorphisms $T_i : O_i \to B_1(0)$ and compact sets $K_1, ..., K_m$ in $\partial\Omega$ such that

(i) $K_i \subset O_i$, $1 \le i \le m$ and $\partial\Omega \subset \cup_{i=1}^m K_i$,

(ii) $T_i(O_i \cap \partial\Omega) = B_1(0) \cap \{x_N = 0\}$, $T_i(O_i \cap \Omega) = B_1(0) \cap \{x_N > 0\}$,

(iii) For any $x \in O_i \cap \Omega$, there exists $y \in O_i \cap \partial\Omega$ such that $\delta(x) = |x-y|$.

We then define the $\mathrm{Cap}^{\partial\Omega}_{\theta,s}$–capacity of a compact set $F \subset \partial\Omega$ by

$$\mathrm{Cap}^{\partial\Omega}_{\theta,s}(F) := \sum_{i=1}^m \mathrm{Cap}_{\mathcal{B}_\theta,s}(\tilde{T}_i(F \cap K_i)), \tag{1.3.39}$$

where $T_i(F \cap K_i) = \tilde{T}_i(F \cap K_i) \times \{x_N = 0\}$.

Let $a > -1$, $0 \le \theta \le \beta < N$ and $s > 1$ and assume that $-1 + s'(1+\theta-\beta) < a < -1 + s'(N+\theta-\beta)$. Then the above capacities are equivalent

$$\mathrm{Cap}^{a}_{\mathbb{N}_{\theta,\beta},s}(E) \approx \mathrm{Cap}^{\partial\Omega}_{\beta-\theta+\frac{a+1}{s'}-1,s}(E) \quad \text{for any Borel } E \subset \partial\Omega.$$

The interested reader is referred to in [78, Section 3.3] for more properties of of such capacities. Using these capacities, we give a criteria for the existence of solutions.

THEOREM 1.14. *Let $\tau \in \mathfrak{M}^+(\Omega,\delta^\alpha)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Assume $p > 1$. Then the following statements are equivalent.*

(i) There exists $C > 0$ such that the following inequalities hold

$$\int_E \delta^\alpha d\tau \le C\, Cap^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2,p'}}(E) \quad \text{for all Borel set } E \subset \overline{\Omega}, \tag{1.3.40}$$

$$\nu(F) \le C\, Cap^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}(F) \quad \text{for all Borel set } F \subset \partial\Omega. \tag{1.3.41}$$

(ii) There exists a positive constant C such that (1.3.31) and (1.3.32) hold.

(iii) Problem (1.3.30) has a positive weak solution for $\sigma > 0$ and $\rho > 0$ small enough.

We remark that capacities are a very useful tool which provides a finer topology than Borel measures. When $1 < p < p_\mu$, we have

$$\inf_{\xi\in\Omega} \mathrm{Cap}^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2,p'}}(\{\xi\}) > 0 \quad \text{and} \quad \inf_{z\in\partial\Omega} \mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}(\{z\}) > 0,$$

hence the Theorem 1.14 covers Theorem 1.13 (i)-(iii).

It is worth mentioning that we also extend existence results for scalar equations to systems of the form

$$\begin{cases} -L_\mu u = \epsilon\, g(v) + \sigma\tau & \text{in } \Omega, \\ -L_\mu v = \epsilon\, \tilde{g}(u) + \tilde{\sigma}\tilde{\tau} & \text{in } \Omega, \\ \mathrm{tr}_\mu(u) = \rho\nu, \quad \mathrm{tr}_\mu(v) = \tilde{\rho}\tilde{\nu} \end{cases} \tag{1.3.42}$$

where g and $\tilde{g}$ are nondecreasing, continuous functions in $\mathbb{R}$ with $g(0) = \tilde{g}(0) = 0$, $\epsilon = \pm 1$, $\sigma > 0$, $\tilde{\sigma} > 0$, $\rho > 0$, $\tilde{\rho} > 0$. The reader is referred to our paper [78] for various existence results for (1.3.42).

1.3.6. Gradient-dependent nonlinearities. In this subsection, we consider the Dirichlet problem for equation (1.2.6) with $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ being nondecreasing and locally Lipschitz in its two variables with $g(0,0) = 0$. We recall that the nonlinearity $g(u,|\nabla u|)$ is called *absorption* (resp. *source*) if the plus sign (resp. minus sign) appears in (1.2.6). Two prototype models to keep in mind are $g(u,|\nabla u|) = u^p + |\nabla u|^q$ and $g(u,|\nabla u|) = u^p|\nabla u|^q$.

First we are interested in the Dirichlet problem in the absorption case

$$\begin{cases} -L_\mu u + g(u,|\nabla u|) = 0 & \text{in } \Omega, \\ \mathrm{tr}_\mu(u) = \nu. \end{cases} \tag{1.3.43}$$

Weak solutions of (1.3.43) are defined in Definition 1.7.

The case where g depends only on $|\nabla u|$ was studied in our joint paper with Gkikas [79] where we showed that the value q_μ given in (1.2.13) is a critical value for the existence of (1.3.43). Several results were established for the Dirichlet problem, including uniqueness and full description of isolated boundary trace in the case $1 < q < q_\mu$ and a removability result in the case $q_\mu \le q < 2$.

Coming back to problem (1.3.43), the existence of a weak solution holds under an integral condition on g (see [80, Theorem 1.3]).

THEOREM 1.15. *Assume g satisfies (1.2.14). Then for any $\nu \in \mathfrak{M}^+(\partial\Omega)$, problem (1.3.43) admits a positive weak solution $0 \le u \le \mathbb{K}_\mu[\nu]$ in Ω.*

The proof is a highly nontrivial adaptation of that in the case where g depends only on u or $|\nabla u|$, relying on sub and super solutions method, the Schauder fixed point theorem and the Vitali convergence theorem.

Two typical models are $g(u, |\nabla u|) = u^p + |\nabla u|^q$ and $g(u, |\nabla u|) = u^p|\nabla u|^q$. The subcritical range and supercritical range for (p, q) are defined in (1.2.15) and (1.2.16).

Next we show that the uniqueness holds in the cases $g(u, |\nabla u|) = u^p + |\nabla u|^q$ and $g(u, |\nabla u|) = u^p|\nabla u|^q$. As a matter of fact, the uniqueness is a direct consequence of the following comparison principle (see [80, Theorem 1.5 and Theorem B.1]). This result is novel even in the case $\mu = 0$.

THEOREM 1.16. *Assume $g(u, |\nabla u|) = u^p + |\nabla u|^q$ with p, q satisfying (1.2.15) or $g(u, |\nabla u|) = u^p|\nabla u|^q$ with p, q satisfying (1.2.16) and $q \ge 1$. Let $\nu_i \in \mathfrak{M}^+(\partial\Omega)$, $i = 1, 2$, and u_i be a nonnegative solution of (1.3.43) with $\nu = \nu_i$. If $\nu_1 \le \nu_2$ then $u_1 \le u_2$ in Ω.*

The proof is based on a regularity result, the maximum principle, estimates on the gradient of subsolutions of a nonhomogeneous linear equation.

Assume the origin $0 \in \partial\Omega$ and let δ_0 be the Dirac measure concentrated at 0. It is known from Theorem 1.15 that for any $k > 0$, there exists a unique solution $u_{0,k}^\Omega$ of (1.3.43) with $\nu = k\delta_0$. It is natural to ask what the $\lim_{k\to\infty} u_{0,k}^\Omega$ could be. The answer is given in the next theorem where a complete description of isolated singularities at 0 is established.

We first consider the case $g(u, |\nabla u|) = u^p + |\nabla u|^q$. In this case, set

$$m_{p,q} := \max\left\{p, \frac{q}{2-q}\right\}. \tag{1.3.44}$$

THEOREM 1.17. *Assume $g(u, |\nabla u|) = u^p + |\nabla u|^q$ with p and q satisfying (1.2.15).*

I. WEAK SINGULARITY. *For any $k > 0$, let $u_{0,k}^\Omega$ be the solution of (1.3.43) with $\nu = k\delta_0$. Then there exists a constant $c = c(N, \mu, \Omega) > 0$ such that*

$$u_{0,k}^\Omega(x) \le ck\delta(x)^\alpha |x|^{2-N-2\alpha} \quad \forall x \in \Omega \tag{1.3.45}$$

and

$$|\nabla u_{0,k}^\Omega(x)| \le ck\delta(x)^{\alpha-1}|x|^{2-N-2\alpha} \quad \forall x \in \Omega. \tag{1.3.46}$$

Moreover

$$\lim_{\Omega \ni x \to 0} \frac{u_{0,k}^\Omega(x)}{K_\mu^\Omega(x, 0)} = k. \tag{1.3.47}$$

Furthermore the mapping $k \mapsto u_{0,k}^\Omega$ is increasing.

II. Strong singularity. *Put* $u^\Omega_{0,\infty} := \lim_{k\to\infty} u^\Omega_{0,k}$. *Then* $u^\Omega_{0,\infty}$ *is a solution of*

$$\begin{cases} -L_\mu u + g(u,|\nabla u|) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{1.3.48}$$

Then there exists a constant $c = c(N,\mu,p,q,\Omega) > 0$ *such that*

$$c^{-1}\delta(x)^\alpha |x|^{-\frac{2}{m_{p,q}-1}} \le u^\Omega_{0,\infty}(x) \le c\delta(x)^\alpha |x|^{-\frac{2}{m_{p,q}-1}} \quad \forall x \in \Omega, \tag{1.3.49}$$

$$|\nabla u^\Omega_{0,\infty}(x)| \le c\delta(x)^{\alpha-1}|x|^{-\frac{2}{m_{p,q}-1}-\alpha} \quad \forall x \in \Omega. \tag{1.3.50}$$

Moreover

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S^{N-1}_+}} |x|^{\frac{2}{m_{p,q}-1}} u^\Omega_{0,\infty}(x) = \omega(\sigma), \tag{1.3.51}$$

locally uniformly on upper hemisphere $S^{N-1}_+ = \mathbb{R}^N_+ \cap S^N$. *Here* $\mathbb{R}^N_+ = \{x = (x_1,...,x_N) = (x',x_N) : x_N > 0\}$ *and* S^{N-1} *is the unit sphere in* $\mathbb{R}^N$. *The function* ω *is the unique positive solution of*

$$\begin{cases} -\mathcal{L}_\mu \omega - \ell_{N,p,q}\omega + J(\omega,\nabla'\omega) = 0 & \text{in } S^{N-1}_+ \\ \omega = 0 & \text{on } \partial S^{N-1}_+ \end{cases} \tag{1.3.52}$$

where

$$\mathcal{L}_\mu \omega = \Delta'\omega + \frac{\mu}{(\mathbf{e}_N\cdot\sigma)^2}\omega, \qquad \ell_{N,p,q} = \frac{2}{m_{p,q}}\Big(\frac{2}{m_{p,q}} + 2 - N\Big),$$

$$J(s,\xi) = \begin{cases} \left(\left(\frac{2}{m_{p,q}}\right)^2 s^2 + |\xi|^2\right)^{\frac{q}{2}}, & \text{if } p < \frac{q}{2-q},\ (s,\xi) \in \mathbb{R}_+\times\mathbb{R}^N \\ s^p + \left(\left(\frac{2}{m_{p,q}}\right)^2 s^2 + |\xi|^2\right)^{\frac{q}{2}}, & \text{if } p = \frac{q}{2-q},\ (s,\xi) \in \mathbb{R}_+\times\mathbb{R}^N \\ s^p, & \text{if } p > \frac{q}{2-q},\ (s,\xi) \in \mathbb{R}_+\times\mathbb{R}^N. \end{cases} \tag{1.3.53}$$

The above theorem shows that there is a competition between two terms u^p and $|\nabla u|^q$. In particular, if $p > \frac{q}{2-q}$ then u^p is the dominant term, otherwise $|\nabla u|^q$ is the dominant term. Moreover, it is observed that the equation is not scaling invariant, unless $p = \frac{q}{2-q}$.

Isolated boundary singularities in the case $g(u,|\nabla u|) = u^p|\nabla u|^q$ are depicted in the next theorem (see [80, Theorem 1.6]). We notice that, unlike the case of sum, in this case of product, the equation is scaling invariant and the blowup rate is explicitly determined by the exponent $\frac{2-q}{p+q-1}$.

Theorem 1.18. *Assume* $g(u,|\nabla u|) = u^p|\nabla u|^q$ *with* $q \ge 1$ *and* p *and* q *satisfying* (1.2.16).

I. Weak singularity. *For any* $k > 0$, *let* $u^\Omega_{0,k}$ *be the solution of* (1.3.43) *with* $\nu = k\delta_0$. *Then* (1.3.45)– (1.3.47) *hold. Furthermore the mapping* $k \mapsto u^\Omega_{0,k}$ *is increasing.*

II. Strong singularity. *Put* $u^\Omega_{0,\infty} := \lim_{k\to\infty} u^\Omega_{0,k}$. *Then* $u^\Omega_{0,\infty}$ *is a solution of* (1.3.48). *There exists a constant* $c = c(N,\mu,p,q,\Omega) > 0$ *such* *that*

$$c^{-1}\delta(x)^{\alpha}|x|^{-\frac{2-q}{p+q-1}-\alpha} \leq u_{0,\infty}^{\Omega}(x) \leq c\delta(x)^{\alpha}|x|^{-\frac{2-q}{p+q-1}-\alpha} \quad \forall x \in \Omega, \tag{1.3.54}$$

$$|\nabla u_{0,\infty}^{\Omega}(x)| \leq c\delta(x)^{\alpha-1}|x|^{-\frac{2-q}{p+q-1}-\alpha} \quad \forall x \in \Omega. \tag{1.3.55}$$

Moreover

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{p+q-1}} u_{0,\infty}^{\Omega}(x) = \omega(\sigma), \tag{1.3.56}$$

locally uniformly on upper hemisphere $S_+^{N-1} = \mathbb{R}_+^N \cap S^{N-1}$, *where* ω *is the unique solution of problem* (1.3.52) *with*

$$\begin{aligned} &\mathcal{L}_\mu \omega = \Delta' \omega + \frac{\mu}{(\mathbf{e}_N \cdot \sigma)^2} w, \qquad \ell_{N,p,q} = \frac{2-q}{p+q-1}\Big(\frac{2p+q}{p+q-1} - N\Big), \\ &J(s,\xi) = s^p \left(\Big(\frac{2-q}{p+q-1}\Big)^2 s^2 + |\xi|^2 \right)^{\frac{q}{2}} \quad (s,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N. \end{aligned} \tag{1.3.57}$$

In the supercritical range, an important ingredient in the study is Bessel capacities. First we recall below some notations concerning Besov spaces and Bessel spaces (see, e.g., [**1, 98, 130**]). For $\sigma > 0$, $1 \leq \kappa < \infty$, we denote by $W^{\sigma,\kappa}(\mathbb{R}^d)$ the Sobolev space over $\mathbb{R}^d$. If σ is not an integer the Besov space $B^{\sigma,\kappa}(\mathbb{R}^d)$ coincides with $W^{\sigma,\kappa}(\mathbb{R}^d)$. When σ is an integer we denote $\Delta_{x,y} f := f(x+y) + f(x-y) - 2f(x)$. The Besov space is defined by

$$B^{1,\kappa}(\mathbb{R}^d) := \left\{ f \in L^\kappa(\mathbb{R}^d) : \frac{\Delta_{x,y} f}{|y|^{1+\frac{d}{\kappa}}} \in L^\kappa(\mathbb{R}^d \times \mathbb{R}^d) \right\},$$

with norm

$$\|f\|_{B^{1,\kappa}} := \left(\|f\|_{L^\kappa}^\kappa + \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|\Delta_{x,y} f|^\kappa}{|y|^{\kappa+d}} dxdy \right)^{\frac{1}{\kappa}}.$$

Then we have

$$B^{m,\kappa}(\mathbb{R}^d) := \left\{ f \in W^{m-1,\kappa}(\mathbb{R}^d) : D_x^\theta f \in B^{1,\kappa}(\mathbb{R}^d) \ \forall \theta \in \mathbb{N}^d, \ |\theta| = m-1 \right\},$$

with norm

$$\|f\|_{B^{m,\kappa}} := \left(\|f\|_{W^{m-1,\kappa}}^\kappa + \sum_{|\theta|=m-1} \int\int_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|D_x^\theta \Delta_{x,y} f|^\kappa}{|y|^{\kappa+d}} dxdy \right)^{\frac{1}{\kappa}}.$$

For $s \in \mathbb{R}$, the Bessel kernel of order s is defined by $G_s(\xi) = \mathcal{F}^{-1}(1 + |.|^2)^{-\frac{s}{2}} \mathcal{F}(\xi)$, where $\mathcal{F}$ is the Fourier transform of moderate distributions in $\mathbb{R}^d$. The Bessel space $L_{s,\kappa}(\mathbb{R}^d)$ is defined by

$$L_{s,\kappa}(\mathbb{R}^d) := \{ f = G_s * g : g \in L^\kappa(\mathbb{R}^d) \},$$

with norm

$$\|f\|_{L_{s,\kappa}} := \|g\|_{L^\kappa} = \|G_{-s} * f\|_{L^\kappa}.$$

It is known that if $1 < \kappa < \infty$ and $s > 0$, $L_{s,\kappa}(\mathbb{R}^d) = W^{s,\kappa}(\mathbb{R}^d)$ if $s \in \mathbb{N}$ and $L_{s,\kappa}(\mathbb{R}^d) = B^{s,\kappa}(\mathbb{R}^d)$ if $s \notin \mathbb{N}$, always with equivalent norms. The Bessel capacity is defined for compact subsets $K \subset \mathbb{R}^d$ by

$$C^{\mathbb{R}^d}_{s,\kappa}(K) := \inf\{\|f\|^{\kappa}_{L_{s,\kappa}}, f \in \mathcal{S}'(\mathbb{R}^d),\ f \geq \chi_K\}. \tag{1.3.58}$$

It is extended to open sets and then Borel sets by the fact that it is an outer measure.

Let $\nu \in \mathfrak{M}^+(\partial\Omega)$. We say that ν is *absolutely continuous* with respect to the Bessel capacity $C^{\mathbb{R}^d}_{s,\kappa}$ if

$$C^{\mathbb{R}^d}_{s,\kappa}(E) = 0 \Longrightarrow \nu(E) = 0 \quad \text{for all Borel set } E.$$

A necessary condition expressed by Bessel capacities for the existence of a solution to (1.3.43) is obtained from [80, Theorem B.5 and Theorem 1.7].

Theorem 1.19. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ and assume that problem (1.3.43) has a nonnegative solution $u \in C^2(\Omega)$.*
I. *Assume $g(u, |\nabla u|) = u^p + |\nabla u|^q$ with (p, q) is in the supercritical range.*
(i) If $p \geq p_\mu$ then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{2-\frac{1+\alpha}{p'},p'}$.
(ii) If $q_\mu \leq q < 2$ then the followings occur.
(a) If $q \neq \alpha+1$ then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{\frac{\alpha+1}{q}-\alpha,q'}$.
(b) If $q = \alpha + 1$ then for any $\varepsilon \in (0, \min\{\alpha + 1, \frac{(N-1)\alpha}{\alpha+1} - (1 - \alpha)\})$ then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{\varepsilon+1-\alpha,\frac{\alpha+1}{\alpha}}$.
II. *Assume $g(u, |\nabla u|) = u^p|\nabla u|^q$ with (p, q) is in the supercritical range.*
(i) If $q \neq \alpha + 1$ then ν is absolutely continuous with respect to the capacity $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}$. Here $(p + q)'$ denotes the conjugate exponent of $p + q$.
(ii) If $q = \alpha + 1$ then for any $\varepsilon \in (0, \min\{\alpha + 1, \frac{(N-1)\alpha}{\alpha+1} - (1 - \alpha)\})$ then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\varepsilon}{p+\alpha+1},(p+\alpha+1)'}$.

Define the weight function W by

$$W(x) := \begin{cases} \delta(x)^{1-\alpha} & \text{if } \mu < \dfrac{1}{4}, \\ \delta(x)^{\frac{1}{2}}|\ln \delta(x)| & \text{if } \mu = \dfrac{1}{4}. \end{cases} \tag{1.3.59}$$

We note that, by [77, Propositions 2.17-2.18], for any $h \in C(\partial\Omega)$ there exists a unique L_μ-harmonic function $u_h \in C(\overline{\Omega}) \cap L^1(\Omega, \delta^\alpha)$ such that

$$\lim_{x\in\Omega,\, x\to\xi} \frac{u_h(x)}{W(x)} = h(\xi) \qquad \forall \xi \in \partial\Omega. \tag{1.3.60}$$

We note that (1.3.60) can be viewed as the boundary condition in the Hardy potential case. If $\mu = 0$ then $\alpha = 1$ and $W(x) \equiv 1$, hence (1.3.60) becomes the boundary condition in the classical sense.

We obtain a removability result in the supercritical range (see [80, Theorem B.6 and Theorem 1.8]).

Theorem 1.20. *Let E be a compact subset of $\partial\Omega$.*
I. *Assume $g(u, |\nabla u|) = u^p + |\nabla u|^q$ with (p, q) is in the supercritical range. Suppose*

(i) $C^{\mathbb{R}^{N-1}}_{2-\frac{1+\alpha}{p'},p'}(E) = 0$ *if* $p \geq p_\mu$,

or (ii) $C^{\mathbb{R}^{N-1}}_{\frac{\alpha+1}{q}-\alpha,q'}(E) = 0$ *if* $q_\mu \leq q < 2$ *and* $q \neq \alpha + 1$.

or (iii) $C^{\mathbb{R}^{N-1}}_{\varepsilon+1-\alpha,q'}(E) = 0$, *for some* $\varepsilon \in (0, \min\{\alpha+1, \frac{(N-1)\alpha}{\alpha+1} - (1-\alpha)\})$ *if* $q = \alpha + 1$.

Then any nonnegative solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus E)$ *of equation*

$$-L_\mu u + g(u, |\nabla u|) = 0 \quad \text{in } \Omega \tag{1.3.61}$$

satisfying

$$\lim_{x\in\Omega,\, x\to\xi} \frac{u(x)}{W(x)} = 0 \qquad \forall \xi \in \partial\Omega \setminus E, \tag{1.3.62}$$

is identically zero.

II. *Assume* $g(u, |\nabla u|) = u^p|\nabla u|^q$ *with* (p,q) *is in the supercritical range. Suppose*

(i) $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(E) = 0$ *if* $q \neq \alpha + 1$,

or (ii) $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\varepsilon}{p+\alpha+1},(p+\alpha+1)'}(E) = 0$, *for some* $\varepsilon \in (0, \min\{\alpha+1, \frac{(N-1)\alpha}{\alpha+1} - (1-\alpha)\})$, *if* $q = \alpha + 1$.

Then any nonnegative solution $u \in C^2(\Omega) \cap C(\overline{\Omega} \setminus E)$ *of equation* (1.3.61) *satisfying* (1.3.62) *is identically zero.*

The results stated in Theorems 1.19 and 1.20 are novel, even for $\mu = 0$.

Next we are interested in the boundary value problem for equations with source term of the form

$$\begin{cases} -L_\mu u - g(u, |\nabla u|) = 0 & \text{in } \Omega, \\ \qquad \operatorname{tr}_\mu(u) = \rho\nu, \end{cases} \tag{1.3.63}$$

where ρ is a positive parameter and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. Weak solutions of (1.3.63) are defined in Definition 1.7.

The source case is different from the absorption case in an essential way. This can be seen in the following result which guarantees the existence of a weak solution *under a smallness assumption on the total mass of the boundary data.*

Theorem 1.21. *Let* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *with* $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. *Assume* g *satisfies* (1.2.14) *and*

$$g(as, bt) \leq \tilde{k}(a^{\tilde{p}} + b^{\tilde{q}})g(s,t) \quad \forall (a,b,s,t) \in \mathbb{R}^4_+, \tag{1.3.64}$$

for some $\tilde{p} > 1$, $\tilde{q} > 1$, $\tilde{k} > 0$. *Then there exists* $\rho_0 > 0$ *depending on* $N, \mu, \Omega, \tilde{k}, \tilde{p}, \tilde{q}$ *such that for any* $\rho \in (0, \rho_0)$, *problem* (1.3.63) *admits a positive weak solution* $u \geq \rho\mathbb{K}_\mu[\nu]$ *in* Ω.

It is easy to see that if $g(u, |\nabla u|) = u^p + |\nabla u|^q$ or $g(u, |\nabla u|) = u^p|\nabla u|^q$ then (1.3.64) holds. Therefore Theorem 1.21 holds true for these typical models.

Sufficient conditions for the existence for the Dirichlet problem in two typical models with (p,q) being in supercritical range are expressed in terms of capacities (see (1.3.37) and (1.3.39)). We note that the capacities used for this case are different from the Bessel capacities employed in the absorption case (see (1.3.58)).

THEOREM 1.22.
I. *Assume* $g(u,|\nabla u|) = u^p + |\nabla u|^q$ *with* $p > 1$ *and* $\frac{\alpha+1}{N+\alpha-1} < q < \frac{1+\alpha}{\alpha}$. *Assume one of the following conditions holds:*

(i) There exists a constant $C > 0$ *such that for every Borel set* $E \subset \partial\Omega$.

$$\nu(E) \le C \min\{\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}(E),\ \mathrm{Cap}^{\partial\Omega}_{-\alpha+\frac{\alpha+1}{q},q'}(E)\}. \tag{1.3.65}$$

(ii) There exists a positive constant $C > 0$

$$\begin{aligned} \mathbb{N}_{2\alpha,2}[\delta^{\alpha(p+1)}\mathbb{N}_{2\alpha,2}[\nu]^p] \le C\mathbb{N}_{2\alpha,2}[\nu] < \infty \quad &\text{a.e. in } \Omega \\ \mathbb{N}_{2\alpha-1,1}[\delta^{(\alpha-1)q+\alpha}\mathbb{N}_{2\alpha-1,1}[\nu]^q] \le C\mathbb{N}_{2\alpha-1,1}[\nu] < \infty \quad &\text{a.e. in } \Omega. \end{aligned} \tag{1.3.66}$$

Then there exists $\rho_0 = \rho_0(N,\mu,p,q,C,\Omega) > 0$ *such that for any* $\rho \in (0,\rho_0)$, *problem* (1.3.63) *admits a weak solution* u.

II. *Assume* $g(u,|\nabla u|) = u^p|\nabla u|^q$ *with* $p \ge 0$, $q \ge 0$, $p+q > 1$ *and* $q < \frac{1+\alpha+(1-\alpha)p}{\alpha}$. *Assume one of the following conditions holds.*

(i) There exists a constant $C > 0$ *such that*

$$\nu(E) \le C\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(E) \quad \text{for every Borel set } E \subset \partial\Omega. \tag{1.3.67}$$

Here $(p+q)'$ *denotes the conjugate exponent of* $p+q$.

(ii) There exists a positive constant $C > 0$ *such that*

$$\mathbb{N}_{2\alpha-1,1}[\delta^{\alpha p+(\alpha-1)q+\alpha}\mathbb{N}_{2\alpha-1,1}[\nu]^{p+q}] \le C\mathbb{N}_{2\alpha-1,1}[\nu] < \infty \quad \text{a.e. in } \Omega. \tag{1.3.68}$$

Then there exists $\rho_0 = \rho_0(N,\mu,p,q,C,\Omega) > 0$ *such that for any* $\rho \in (0,\rho_0)$, *problem* (1.3.63) *admits a weak solution* u.

1.4. Related and open problems

We notice that because of the broadness of the topics regarding boundary value problems for nonlinear equations with a Hardy potential, the results presented above are a modest contribution to the recent developments and are due to the our interest. The topics have received much attention and many new results have been recently published. Some related interesting results and open problems are listed below.

Multplicity and uniqueness. As pointed out before in Theorem 1.12, when $1 < p < p_\mu$, there exists a threshold value $\rho^* > 0$ such that problem (1.3.29) admits a minimal positive solution for $\rho \in (0,\rho^*]$ and admits no positive weak solution for $\rho > \rho^*$. In [**23**], we carried out a deeper analysis on (1.3.29) and proved the multiplicity for $\rho \in (0,\rho^*)$ and the uniqueness for $\rho = \rho^*$, which complements the results in [**119**]. More precisely, the structure of the solution set of problem (1.3.29) is described as follows.

Subcritical case: $p \in (1,p_\mu)$. There exists $\rho^* \in (0,\infty)$ such that the followings hold.

(i) If $\rho \in (0,\rho^*)$ then problem (1.3.26) admits two positive solutions, including the minimal positive solution.
(ii) If $\rho = \rho^*$ then problem (1.3.26) admits a unique positive solution.
(iii) If $\rho > \rho^*$ then (1.3.29) does not admit any positive solution.

Supercritical case: $p \geq p_\mu$. For every $\rho > 0$ and $z \in \partial\Omega$, there is no positive weak solution of (1.3.29) with $\nu = \delta_y$, where δ_y is the Dirac mass concentrated at $y \in \partial\Omega$.

The multiplicity for systems were also derived in [**23**] for measure data with small total mass. However, this result provides partial understanding of solution set of systems and needs to be improved.

Schrödinger equations with potential blowing up on boundary. Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a C^2 bounded domain and $F \subset \partial\Omega$ be a k dimensional C^2 submanifold, $0 \leq k \leq N-1$. Denote

$$\delta(x) := \mathrm{dist}(x, \partial\Omega), \quad \delta_F(x) := \mathrm{dist}(x, F)$$

and put

$$V(x) = V_F(x) := \delta_F(x)^{-2}, \quad x \in \Omega.$$

Boundary value problems for semilinear equations with Schrödinger operator $L_V := \Delta + \mu V$, $\mu \in \mathbb{R}$, in the special cases $k = N$ and $k = 0$ have been extensively investigated. In particular, the case $k = N-1$, $F = \partial\Omega$ and $V(x) = \delta(x)^{-2}$ is well understood, as shown in this thesis, while the case $k = 0$, $F = \{0\}$ with the origin $0 \in \partial\Omega$ and $V(x) = |x|^{-2}$ was treated by Chen and Véron [**45**]. It is also worth mentioning that the case of more strongly singular potential $V(x) = \delta(x)^{-\alpha}$ with $\alpha > 2$ was considered by Du and Wei in [**54**]. The case $1 \leq k \leq N-2$ had remained open until our work in [**107**]. In fact, we considered the case where F is a C^2 submanifold of dimension $0 \leq k \leq N-2$ without boundary and established the solvability for solutions to the equation $-L_V u + g(u) = 0$ in Ω with prescribed boundary data. The reader is referred to the work of Fall and Mahmoudi [**65**] for Hardy type estimates and estimate of the first eigenfunction.

Recently, Marcus published papers [**100, 101**] in which he considered the potential V such that $|V(x)| \leq a\delta(x)^{-2}$ for all $x \in \Omega$ and under additional conditions, he obtained estimates on Green kernel, Martin kernel, as well as sub and super harmonic functions. Moreover, large solutions of semilinear equations are studied in [**102**].

Schrödinger equations with potential blowing up on a subset of the domain Another interesting case is that

$$V(x) = \mu\delta_\Sigma(x)^{-2}, \tag{1.4.1}$$

where Σ is a compact, C^2 submanifold in Ω with dimension k with $0 \leq k < N-2$ and $\delta_\Sigma(x) = \mathrm{dist}(x, \Sigma)$. The special case $\Sigma = \{0\} \subset \Omega$ was treated by Guerch and Véron [**83**], Chen and Véron [**44**], Chen and Zhou [**46**], Cîrstea [**47**] and references therein. The case V satisfies (1.4.1) was considered in a series of papers of Dávila and Dupaigne [**49, 50, 55**] where linear and nonlinear equations involving $L_V = \Delta + V$ with source term was studied. Recently, a complete study on the Green kernel and Martin kernel was given in our joint paper with Gkikas [**81**] which provides a basis in dealing with boundary value problems for linear equations regarding L_V in a different framework. In this direction, the analysis of semilinear equations involving L_V and absorption term is more challenging and still open.

Bibliography

[1] D. R. Adams, L.I. Heberg, *Function Spaces and Potential Theory,* Grundlehren der Mathematischen Wisenschaften 31, Springer-Verlag (1999).

[2] A. Aghajani and C. Cowan, *Explicit estimates on positive supersolutions of nonlinear elliptic equations and applications*, Discrete Contin. Dyn. Syst. **39** (2019), no. 5, 2731–2742.

[3] A. Aghajani, C. Cowan and S. H. Lui, *Existence and regularity of nonlinear advection problems*, Nonlinear Anal. **166** (2018), 19–47.

[4] A. Aghajani, C. Cowan and S. H. Lui, *Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball*, J. Differential Equations **264** (2018), no. 4, 2865–2896.

[5] S. Alarcón, J. García-Melián and A. Quaas, *Nonexistence of positive supersolutions to some nonlinear elliptic problems*, J. Math. Pures Appl. (9) **99** (2013), no. 5, 618–634.

[6] S. Alarcón, J. García-Melián and A. Quaas, *Keller-Osserman type conditions for some elliptic problems with gradient terms*, J. Differential Equations **252** (2012) 886–914.

[7] A. Ancona, *Theorié du potentiel sur les graphes et les variétés*, in Ecole d'été de Probabilités de Saint-Flour XVIII-1988, Springer Lecture Notes in Math, **1427** (1990), 1-112.

[8] A. Ancona, *Negatively curved manifolds, elliptic operators and the Martin boundary*, Ann. of Math. (2), **125** (1987), 495-536.

[9] A. Ancona and M. Marcus, *Positive solutions of a class of semilinear equations with absorption and Schrödinger equations*, J. Math. Pures Appl. **104** (2015), 587-618.

[10] C. Bandle, M. Marcus and V. Moroz, *Boundary singularities of solutions of semilinear elliptic equations in the half-space with a Hardy potential*, Israel J. Math. **222** (2017), no. 1, 487-514.

[11] C. Bandle, V. Moroz and W. Reichel, *Boundary blowup type sub-solutions to semilinear elliptic equations with Hardy potential*, J. London Math. Soc. **2** (2008), 503-523.

[12] C. Bandle and M. A. Pozio, *Sublinear elliptic problems with a Hardy potential*, Nonlinear Anal. **119** (2015), 149–166.

[13] C. Bandle and M. A. Pozio, *Positive solutions of semilinear elliptic problems with a Hardy potential*, Proc. Roy. Soc. Edinburgh Sect. A **150** (2020), no. 4, 1791–1804.

[14] P. Baras and M. Pierre, *Singularités éliminables pour des équations semi-linéaires*, Ann. Inst. Fourier (Grenoble) **34** (1984), no. 1, 185–206.

[15] P. Baras and M. Pierre, *Critères dexistence de solutions positives pour des équations semi-linéaires non monotones*, Ann. Inst. H. Poincaré, Analyse Non Linéaire **2** (1985), 185–212.

[16] G. Barbatis, S. Filippas and A. Tertikas, *A unified approach to improved L^p Hardy inequalities with best constants,* Trans. Amer. Math. Soc. **356** (2003) 2169-2196.

[17] R. D. Benguria, *The Lane-Emden equation revisited* (English summary), Advances in differential equations and mathematical physics (Birmingham, AL, 2002), 11–19, Contemp. Math., **327**, Amer. Math. Soc., Providence, RI, 2003.

[18] R. Benguria, H. Brezis and E. H. Lieb, *The Thomas-Fermi-von Weizscker theory of atoms and molecules*, Comm. Math. Phys. **79** (1981), no. 2, 167–180.

[19] R. Benguria and L. Jeanneret, *Existence and uniqueness of positive solutions of semilinear elliptic equations with Coulomb potentials on* $\mathbb{R}^3$, Comm. Math. Phys. **104** (1986), no. 2, 291–306.

[20] P. Bénilan and H. Brezis, *Nonlinear problems related to the Thomas-Fermi equation. Dedicated to Philippe Bénilan*, J. Evol. Equ. **3** (2003), no. 4, 673–770.

[21] P. Bénilan, H. Brezis and M. G. Crandall, *A semilinear equation in* $L^1(\mathbb{R}^N)$, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **2** (1975), no. 4, 523–555.

[22] M. Bhakta and M. Marcus, *Semilinear elliptic equations admitting similarity transformations*, J. Funct. Anal. **267** (2014), no. 10, 3894–3930.

[23] M. Bhakta, Debangana Mukherjee and P.-T. Nguyen, *Multiplicity and uniqueness for Lane-Emden equations and systems with Hardy potential and measure data*, preprint (2020).

[24] M. F. Bidaut-Véron, R. Borghol and L. Véron, *Boundary Harnack inequality and a priori estimates of singular solutions of quasilinear elliptic equations*, Calc. Var. Partial Differential Equations **27** (2006), no. 2, 159–177.

[25] M.-F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, *Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 3471–3515.

[26] M.-F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, *Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient*, Duke Math. J. **168** (2019), no. 8, 1487–1537.

[27] M.-F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, *A priori estimates for elliptic equations with reaction terms involving the function and its gradient*, Math. Ann. **378** (2020), no. 1-2, 1356.

[28] M. F. Bidaut-Véron, G. Hoang, Q. H. Nguyen and L. Véron, *An elliptic semilinear equation with source term and boundary measure data: the supercritical case*, Journal of Functional Analysis **269** (2015), 1995-2017.

[29] M. F. Bidaut-Véron, M. G. Huidobro, L. Véron, *Laurent Boundary singularities of positive solutions of quasilinear Hamilton-Jacobi equations*, Calc. Var. Partial Differential Equations **54** (2015), no. 4, 3471–3515.

[30] M. F. Bidaut-Véron, A. C. Ponce and L. Véron, *Isolated boundary singularities of semilinear elliptic equations*, Calc. Var. Partial Differential Equations **40** (2011), no. 1-2, 183–221.

[31] M. F. Bidaut-Véron and L. Vivier, *An elliptic semilinear equation with source term involving boundary measures: the subcritical case*, Rev. Mat. Iberoamericana **16**, 477–513 (2000).

[32] M. F. Bidaut-Véron and C. Yarur, *Semilinear elliptic equations and systems with measure data: existence and a priori estimates*, Adv. Differ. Equ. **7**, 257–296 (2002).

[33] H. Brezis, *Une équation semilinéaire avec conditions aux limites dans* L^1, Unpublished note, (1972).

[34] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext. Springer, New York (2011), xiv+599 pp.

[35] H. Brezis and F. Browder, *Partial differential equations in the 20th century*, Adv. Math. **135** (1998), no. 1, 76–144.

[36] H. Brezis and M. Marcus, *Hardy inequalities revisited*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25** (1997) 217-237.

[37] H. Brezis, M. Marcus and A. C. Ponce, *Nonlinear elliptic equations with measures revisited*. In: Mathematical Aspects of Nonlinear Dispersive Equations (J. Bourgain, C. Kenig, S. Klainerman, eds.), Annals of Mathematics Studies, **163**, Princeton University Press, Princeton, NJ, 2007, pp. 55–110.

[38] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36** (1983), no. 4, 437–477.

[39] H. Brezis and L. Oswald, *Singular solutions for some semilinear elliptic equations*, Arch. Rat. Mech. Anal. **99** (1987), 249–259.

[40] H. Brezis and W. Strauss, *Semilinear elliptic equations in* L^1, J. Math. Soc. Japan **25** (1973), 565-590.

[41] H. Brezis and L. Véron, *Removable singularities for some nonlinear elliptic equations*, Arch. Rational Mech. Anal. **75** (1980/81), no. 1, 1-6.

[42] S. Chandrasekhar, *An Introduction to the Study of Stellar Structure*, Dover Publications Inc., NY, 1967.

[43] H. Chen, P. Felmer and L. Véron, *Elliptic equations involving general subcritical source nonlinearity and measures*, arxiv:1409.3067 (2014).

[44] H. Chen and L. Véron, *Weak solutions of semilinear elliptic equations with Leray-Hardy potentials and measure data*, Math. Eng. **1** (2019), no. 3, 391–418.

[45] H. Chen and L. Véron, *Laurent Schrdinger operators with Leray-Hardy potential singular on the boundary*, J. Differential Equations **269** (2020), no. 3, 2091–2131.

[46] H. Chen and F. Zhou, *Isolated singularities for elliptic equations with Hardy operator and source nonlinearity*, Discrete Contin. Dyn. Syst. **38** (2018), no. 6, 2945–2964.

[47] F. C. Cîrstea, *A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials*, Mem. Amer. Math. Soc. **227** (2014), no. 1068, vi+85 pp.

[48] J. Ching and F. Cîrstea, *Existence and classification of singular solutions to nonlinear elliptic equations with a gradient term*, Anal. PDE **8** (2015), no. 8, 1931–1962.

[49] J.Dávila and L. Dupaigne, *Comparison result for PDE's with a singular potential*, Proc. Roy. Soc. Edinburgh Sect. A **133** (2003), 61-83.

[50] J.Dávila and L. Dupaigne, *Hardy-type inequalities,* J. Eur. Math. Soc. **6** (2004), 335-365.

[51] P. Drábek and J. Milota, Methods of nonlinear analysis. Applications to differential equations, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2007. xii+568 pp.

[52] Y. Du and L. Wei, *Boundary behavior of positive solutions to nonlinear elliptic equations with Hardy potential*, J. Lond. Math. Soc. (2) **91** (2015), no. 3, 731–749.

[53] Y. Du and L. Wei, *Exact singular behavior of positive solutions to nonlinear elliptic equations with a Hardy potential*, J. Differential Equations **262** (2017), no. 7, 3864–3886.

[54] Y. Du and L. Wei, *Positive solutions of elliptic equations with a strong singular potential*, Bull. Lond. Math. Soc. **51** (2019), no. 2, 251–266.

[55] L. Dupaigne, *A nonlinear elliptic PDE with the inverse square potential*, J. Anal. Math. **86** (2002), 359–398.

[56] E. B. Dynkin, *A probabilistic approach to one class of nonlinear differential equations*, Probab. Theory Related Fields **89** (1991), 89-115.

[57] E. B. Dynkin, *Diffusions, Superdiffusions and Partial Differential Equations*, Amer. Math. Soc., Providence, RI, 2002.

[58] E. B. Dynkin, *Superdiffusions and positive solutions of nonlinear partial differential equations*, University Lecture Series **34**. Amer. Math. Soc., Providence, vi+120 pp (2004).

[59] E. B. Dynkin and S. E. Kuznetsov, *Superdiffusions and removable singularities for quasilinear partial differential equations*, Comm. Pure Appl. Math. **49** (1996), 125-176.

[60] E. B. Dynkin and S. E. Kuznetsov, *Solutions of $Lu = u^\alpha$ dominated by harmonic functions*, J. Analyse Math. **68** (1996), 15-37.

[61] E. B. Dynkin and S. E. Kuznetsov, *Fine topology and fine trace on the boundary associated with a class of semilinear differential equations*, Comm. Pure Appl. Math. **51** (1998), 897-936.

[62] E. B. Dynkin and S. E. Kuznetsov, *Trace on the boundary for solutions of nonlinear differential equations*, Trans. Amer. Math. Soc. **350** (1998), no. 11, 4499–4519.

[63] L. C. Evans, *Partial differential equations.* Second edition. Graduate Studies in Mathematics, 19. American Mathematical Society, Providence, RI, 2010. xxii+749 pp.

[64] L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions.* Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1992. viii+268 pp.

[65] M. Fall and F. Mahmoudi, *Weighted Hardy inequality with higher dimensional singularity on the boundary*, Calc. Var. Partial Differential Equations **50** (2014), no. 3-4, 779–798.

[66] S. Filippas, L. Moschini and A. Tertikas, *Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains*, Commun. Math. Phys. **273** (2007), 237–281.

[67] R. Filippucci, P. Pucci and P. Souplet, *A Liouville-type theorem in a half-space and its applications to the gradient blow-up behavior for superquadratic diffusive Hamilton-Jacobi equations*, Comm. Partial Differential Equations **45** (2020), no. 4, 321–349.

[68] R. Filippucci, P. Pucci and P. Souplet, *A Liouville-type theorem for an elliptic equation with superquadratic growth in the gradient*, Adv. Nonlinear Stud. **20** (2020), no. 2, 245–251.

[69] R. Frank, *Sobolev inequalities and uncertainty principles in mathematical physics. part I*, unpublished notes of a course given at the LMU, Munich. Available From: http://www.math.caltech.edu/ rlfrank/sobweb1.pdf.

[70] A. Friedman and L. Véron, *Singular solutions of some quasilinear elliptic equations*, Arch. Rational Mech. Anal. **96** (1986), no. 4, 359–387.

[71] M. Ghergu and V. Rădulescu, *On a class of sublinear singular elliptic problems with convection term*, J. Math.Anal.Appl. **311** (2005), 635–646.

[72] M. Ghergu and V. Rădulescu, *Ground state solutions for the singular Lane-Emden-Fowler equation with sublinear convection term*, J. Math. Anal. Appl. **333** (2007), no. 1, 265–273.

[73] B. Gidas, W.M. Ni and L. Nirenberg, *Symmetry and related properties via the maximum principle*, Comm. Math. Phys. **68** (3) (1979), 209–243.

[74] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (8) (1981), 883–901.

[75] B. Gidas and J. Spruck, *Global and local behavior of positive solutions of nonlinear elliptic equations*, Comm. Pure Appl. Math. **34** (4) (1981), 525–598.

[76] D. Gilbarg and N. Trudinger, *Elliptic Partial Differential Equations of Second Order.* (Second edition), Springer, Berlin (1983).

[77] K. T. Gkikas and L. Véron, *Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials*, Nonlinear Anal. **121** (2015), 469-540.

[78] K. T. Gkikas and P. T. Nguyen, *On the existence of weak solutions of semilinear elliptic equations and systems with Hardy potentials*, J. Differential Equations **266** (2019), no. 1, 833–875.

[79] K. Gkikas and P. T. Nguyen, *Semilinear elliptic equations with Hardy potential and gradient nonlinearity*, Rev. Mat. Iberoam. **36** (2020), no. 4, 1207–1256.

[80] K. T. Gkikas and P. T. Nguyen, *Elliptic equations with Hardy potential and gradient-dependent nonlinearity*, Adv. Nonlinear Stud. **20** (2020), no. 2, 399-435.

[81] K. T. Gkikas and P. T. Nguyen, *Green kernel and Martin kernel of Schrdinger operators with singular potential and application to the B.V.P. for linear elliptic equations*, preprint (2020), https://arxiv.org/abs/2002.10754.

[82] A. Gmira and L. Véron, *Boundary singularities of solutions of some nonlinear elliptic equations*, Duke Math. J. **64** (1991), 271-324.

[83] B. Guerch and L. Véron, *Local properties of stationary solutions of some nonlinear singular Schrödinger equations*, Rev. Mat. Iberoamericana 7 (1991), no. 1, 65–114.

[84] G. H. Hardy, *Note on a Theorem of Hilbert*, Math. **6** (1920), 314–317.

[85] G. H. Hardy, *An inequality between integrals*, Messenger of Math. **54** (1925), 150–156.

[86] N. J. Kalton and I. V. Verbitsky, *Nonlinear equations and weighted norm inequalities*, Trans. A. M. S., **351** (1999), 3341–3397.

[87] J.B. Keller, *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503-510.

[88] O.A. Ladyzhenskaya and N. N. Uraltseva, *Linear and quasilinear elliptic equations.* Translated from the Russian by Scripta Technica, Inc. Translation editor: Leon Ehrenpreis Academic Press, New York-London (1968).

[89] J. H. Lane, *On the theoretical temperature of the Sun, under the hypothesis of a gaseous mass maintaining its volume by its internal heat, and depending on the laws of gases as known to terrestrial experiment*, Am. J. Sci., 2d ser. **50** (1869), 57–74.

[90] J.M. Lasry, P.L. Lions, *Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem*, Math. Ann. **283** (1989) 583–630.

[91] J. F. Le Gall, *Les solutions positives de $\Delta u = u^2$ dans le disque unité*, C.R. Acad. Sci. Paris Sér. I **317**, (1993), 873–878.

[92] J. F. Le Gall, *The Brownian snake and solutions of $\Delta u = u^2$ in a domain*, Probab. Th. Rel. Fields **102** (1995), 393–432.

[93] J. F. Le Gall, *A probabilistic Poisson representation for positive solutions of $\Delta u = u^2$ in a domain*, Comm. Pure Appl. Math. **50** (1997), 69–103.

[94] J. F. Le Gall, *A probabilistic approach to the trace at the boundary for solutions of a semilinear parabolic partial differential equation*, J. Appl. Math. Stochastic Anal. **9** (1996), 399-414.

[95] E. H. Lieb, *A brief review of Thomas-Fermi theory*, available at https://physics.nyu.edu/LarrySpruch/Lieb.pdf

[96] P.L. Lions, *Isolated singularities in semilinear problems*, Journal. Diff. Eq., **38** (1980), 441–450.

[97] P.L. Lions, *Quelques remarques sur les problème elliptiques quasilineaires du second ordre*, J. Anal. Math. **45** (1985), 234–254.

[98] J. Malý and W. P. Ziemer, *Fine regularity of solutions of elliptic partial differential equations. Mathematical Surveys and Monographs*, 51. American Mathematical Society, Providence, RI, 1997. xiv+291 pp.

[99] M. Marcus, *Complete classification of the positive solutionsof $-\Delta u + u^q = 0$*, J. d'Anal. Math. **117** (2012), 187–220.

[100] M. Marcus, *Estimates of Green and Martin kernels for Schrödinger operators with singular potential in Lipschitz domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **36** (2019), no. 5, 1183-1200.

[101] M. Marcus, *Estimates of sub and super solutions of Schrödinger equations with very singular potentials*, preprint (2019), https://arxiv.org/abs/1912.01283.

[102] M. Marcus, *Large solutions of semilinear equations with Hardy potential*, preprint (2020), https://arxiv.org/abs/2006.02993.

[103] M. Marcus, V. J. Mizel and Y. Pinchover, *On the best constant for Hardy's inequality in $\mathbb{R}^N$*, Trans. Amer. Math. Soc. **350** (1998), 3237-3255.

[104] M. Marcus and V. Moroz, *Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **18** (2018), no. 1, 3964.

[105] M. Marcus and P.-T. Nguyen, *Phuoc-Tai Elliptic equations with nonlinear absorption depending on the solution and its gradient*, Proc. Lond. Math. Soc. (3) **111** (2015), no. 1, 205–239.

[106] M. Marcus and P.-T. Nguyen, *Moderate solutions of semilinear elliptic equations with Hardy potential*, Ann. Inst. H. Poincaré Anal. Non Linéaire **34** (2017), no. 1, 69-88.

[107] M. Marcus and P. T. Nguyen, *Schrödinger equations with singular potentials: linear and nonlinear boundary value problems*, Math. Ann. **374** (2019), no. 1-2, 361-394.

[108] M. Marcus and I. Shafrir, *An eigenvalue problem related to Hardy's L^p inequality*, Ann. Scuola. Norm. Sup. Pisa. Cl. Sc. **29** (2000), 581-604.

[109] M. Marcus and L. Véron, *Trace au bord des solutions positives d'équations elliptiques non linéaires*, C. R. Acad. Sci. Paris Sér. I Math. **321** (1995), no. 2, 179–184.

[110] M. Marcus and L. Véron, *Trace au bord des solutions positives d'équations elliptiques et paraboliques non linéaires. Résultats d'existence et d'unicité*, C. R. Acad. Sci. Paris Sér. I Math. **323** (1996), no. 6, 603–608.

[111] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case*, Arch. Rational Mech. Anal. **144** (1998), 201-231.

[112] M. Marcus and L. Véron, *The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case*, J. Math. Pures Appl. (9) **77** (1998), 481-524.
[113] M. Marcus and L. Véron, *Removable singularities and boundary trace*, J. Math. Pures Appl. **80** (2001), 879-900.
[114] M. Marcus and L. Véron, *Capacitary estimates of positive solutions of semilinear elliptic equations with absorbtion*, J. Eur. Math. Soc. **6** (2004), 483-527.
[115] M. Marcus and L. Véron, *The boundary trace and generalized boundary value problem for semilinear elliptic equations with coercive absorption*, Comm. Pure Appl. Math. **LVI** (2003), 689-731.
[116] M. Marcus and L. Véron, *Nonlinear second order elliptic equations involving measures*, De Gruyter Series in Nonlinear Analysis and Applications, 2013.
[117] B. Mselati, *Classification and probabilistic representation of the positive solutions of a semilinear elliptic equations*, Mem. Amer. Math. Soc. **168** (2004), no. 798.
[118] J. Necas, *Sur une méthode pour résoudre les équations aux dérivées partielles du type elliptique, voisine de la variationelle*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. **16** (1962), 305–326.
[119] P.-T. Nguyen, *Semilinear elliptic equations with Hardy potential and subcritical source term*, Calc. Var. Partial Differential Equations **56** (2017), no. 2, Paper No. 44, 28 pp.
[120] P.-T. Nguyen and L. Véron, *Boundary singularities of solutions to elliptic viscous Hamilton-Jacobi equations*, J. Funct. Anal. **263** (2012), no. 6, 1487–1538.
[121] B. Opic and A. Kufner, *Hardy-type Inequalities*, Pitman Research Notes in Math., Vol. 219, Longman, 1990.
[122] L. Orsina and A. Ponce, *Strong maximum principle for Schrödinger operators with singular potential*, Ann. Inst. H. Poincaré Anal. Non Linéaire **33** (2016), no. 2, 477–493.
[123] L. Orsina and A. Ponce, *Hopf potentials for the Schrödinger operator*, Anal. PDE **11** (2018), no. 8, 2015–2047.
[124] R. Osserman, *On the inequality $\Delta u \geq f(u)$*, Pacific J. Math. **7** (1957), 1641-1647.
[125] P. Poláčik, P. Quittner and P. Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems*, Duke Math. J. **139** (2007), no. 3, 555–579.
[126] A. C. Ponce and N. Wilmet, *Schrödinger operators involving singular potentials and measure data*, J. Differential Equations **263** (2017), no. 6, 3581–3610.
[127] P. Quittner and P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser Verlag, Basel, 2007. xii+584 pp.
[128] P. Quittner and P. Souplet, *Superlinear parabolic problems. Blow-up, global existence and steady states. Second edition of [MR2346798]*, Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks] Birkhäuser/Springer, Cham, 2019. xvi+725 pp.
[129] B. Simon, *Schrödinger operators in the twentieth century*, J. Math. Phys. **41** (2000), no. 6, 3523–3555.
[130] E. M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
[131] Terence Tao, *The Schrödinger equation*, https://www.math.ucla.edu/ tao/preprints/schrodinger.pdf.
[132] N. Trudinger, *Local estimates for subsolutions and supersolutions of general second order elliptic quasilinear equations*, Invent. Math. **61** (1980), 67-79.
[133] J. L. Vázquez and L. Véron, *Laurent Removable singularities of some strongly nonlinear elliptic equations*, Manuscripta Math. **33** (1980/81), no. 2, 129–144.
[134] J. L. Vázquez and L. Véron, *Singularities of elliptic equations with an exponential nonlinearity*, Math. Ann. **269** (1984), no. 1, 119–135.
[135] J. L. Vázquez and L. Véron, *Isolated singularities of some semilinear elliptic equations*, J. Differential Equations **60** (1985), no. 3, 301–321.
[136] L. Véron, *Singular solutions of some nonlinear elliptic equations*, Nonlinear Anal. **5** (1981), no. 3, 225–242.

[137] L. Véron, *Laurent Singularités éliminables d'équations elliptiques non linéaires* (French), J. Differential Equations **41** (1981), no. 1, 87–95.

[138] L. Véron, *Singularities of solutions of second order quasilinear equations.* Pitman Research Notes in Mathematics Series, 353. Longman, Harlow, 1996. viii+377 pp.

[139] L. Véron, *Elliptic equations involving measures. Stationary partial differential equations.* Vol. I, 593712, Handb. Differ. Equ., North-Holland, Amsterdam, 2004.

[140] L. Véron, *Laurent Local and global aspects of quasilinear degenerate elliptic equations. Quasilinear elliptic singular problems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017. xv+457 pp.

[141] L. Véron and C. Yarur, *Boundary value problems with measures for elliptic equations with singular potentials*, Appendix A by Alano Ancona, J. Funct. Anal. **262** (2012), no. 3, 733–772.

CHAPTER 2

Moderate solutions of semilinear elliptic equations with a Hardy potential

This chapter is based on our join paper with Moshe Marcus [**106**] on boundary value problems for semilinear elliptic equations with an absorption term and a Hardy potential. In this chapter, we introduce a notion of normalized boundary trace and develop a theory of linear equations, which in turn provides a basis for the study of semilinear equations.

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Moderate solutions of semilinear elliptic equations with Hardy potential

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Abstract

Let Ω be a bounded smooth domain in $\mathbb{R}^N$. We study positive solutions of equation (E) $-L_\mu u+u^q=0$ in Ω where $L_\mu=\Delta+\frac{\mu}{\delta^2}$, $0<\mu$, $q>1$ and $\delta(x)=\operatorname{dist}(x,\partial\Omega)$. A positive solution of (E) is moderate if it is dominated by an L_μ-harmonic function. If $\mu<C_H(\Omega)$ (the Hardy constant for Ω) every positive L_μ-harmonic function can be represented in terms of a finite measure on $\partial\Omega$ via the Martin representation theorem. However the classical measure boundary trace of any such solution is zero. We introduce a notion of normalized boundary trace by which we obtain a complete classification of the positive moderate solutions of (E) in the subcritical case, $1<q<q_{\mu,c}$. (The critical value depends only on N and μ.) For $q\ge q_{\mu,c}$ there exists no moderate solution with an isolated singularity on the boundary. The normalized boundary trace and associated boundary value problems are also discussed in detail for the linear operator L_μ. These results form the basis for the study of the nonlinear problem.

MSC: 35J60; 35J75; 35J10

Keywords: Hardy potential; Martin kernel; Moderate solutions; Normalized boundary trace; Critical exponent; Removable singularities

1. Introduction

In this paper, we investigate boundary value problem with measure data for the following equation

$$-\Delta u-\frac{\mu}{\delta^2}u+u^q=0 \tag{1.1}$$

in a C^2 bounded domain Ω, where $q>1$, $\mu\in\mathbb{R}$ and $\delta(x)=\operatorname{dist}(x,\partial\Omega)$. This problem is naturally linked to the theory of linear Schrödinger equations $-L^V u=0$ where $L^V:=\Delta+V$ and the potential V satisfies $|V|\le c\delta^{-2}$. Such equations have been studied in numerous papers (see [1,2] and the references therein).

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Put

$$L_\mu := \Delta + \frac{\mu}{\delta^2}. \tag{1.2}$$

A solution $u \in L^1_{loc}(\Omega)$ of the equation $-L_\mu u = 0$ is called an L_μ*-harmonic function*. Similarly, if

$$-L_\mu u \geq 0 \quad \text{or} \quad -L_\mu u \leq 0$$

we say that u is L_μ*-superharmonic* or L_μ*-subharmonic* respectively. If $\mu = 0$ we shall just use the terms harmonic, superharmonic, subharmonic.

Some problems involving equations (1.1) and (1.2) with $\mu < 1/4$ were studied by Bandle, Moroz and Reichel [4]. They derived estimates of local L_μ-subharmonic and superharmonic functions and applied these results to study conditions for existence or nonexistence of large solutions of (1.1). They also showed that the classical Keller–Osserman estimate [14,24] remains valid for (1.1).

The condition $\mu < \frac{1}{4}$ is related to Hardy's inequality. Denote by $C_H(\Omega)$ the best constant in Hardy's inequality, i.e.,

$$C_H(\Omega) = \inf_{H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega (u/\delta)^2 dx}. \tag{1.3}$$

By Marcus, Mizel and Pinchover [17], $C_H(\Omega) \in (0, \frac{1}{4}]$ and $C_H(\Omega) = \frac{1}{4}$ when Ω is convex. Furthermore the infimum is achieved if and only if $C_H(\Omega) < 1/4$. By Brezis and Marcus [7], for every $\mu < 1/4$ there exists a unique number $\lambda_{\mu,1}$ such that

$$\mu = \inf_{H^1_0(\Omega)} \frac{\int_\Omega (|\nabla u|^2 - \lambda_{\mu,1} u^2) dx}{\int_\Omega (u/\delta)^2 dx}$$

and the infimum is achieved. Thus $\lambda_{\mu,1}$ is an eigenvalue of $-L_\mu$ and, by [7, Lemma 2.1], it is a simple eigenvalue. We denote by $\varphi_{\mu,1}$ the corresponding positive eigenfunction normalized by $\int_\Omega (\varphi^2_{\mu,1}/\delta^2) dx = 1$.

The mapping $[1/4, \infty) \ni \mu \mapsto \lambda_{\mu,1}$ is strictly decreasing. Therefore if $\mu < C_H(\Omega)$ then $\lambda_{\mu,1} > 0$. Consequently, in this case, $\varphi_{\mu,1}$ is a positive supersolution of $-L_\mu$. This fact and a classical result of Ancona [2] imply that for every $y \in \partial\Omega$, there exists a positive L_μ-harmonic function in Ω which vanishes on $\partial\Omega \setminus \{y\}$ and is unique up to a constant. Denote this function by $K^\Omega_\mu(\cdot, y)$, normalized by setting it equal to 1 at a fixed reference point $x_0 \in \Omega$. The function $(x, y) \mapsto K^\Omega_\mu(x, y)$, $(x, y) \in \Omega \times \partial\Omega$, is the L_μ*-Martin kernel* in Ω relative to x_0. Further, by [2]:

Representation Theorem. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ the function*

$$\mathbb{K}^\Omega_\mu[\nu](x) := \int_{\partial\Omega} K^\Omega_\mu(x, y) d\nu(y) \quad \forall x \in \Omega \tag{1.4}$$

is L_μ-harmonic, i.e., $L_\mu \mathbb{K}^\Omega_\mu[\nu] = 0$. Conversely, for every positive L_μ-harmonic function u there exists a unique measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $u = \mathbb{K}^\Omega_\mu[\nu]$.

This theorem implies that – in the present case – the L_μ-Martin boundary of Ω coincides with the Euclidean boundary. (For the general definition of Martin boundary see, e.g. [1]. However this notion will not be used here beyond the representation theorem stated above.) The measure ν such that $u = \mathbb{K}^\Omega_\mu[\nu]$ is called the L_μ*-boundary measure of* u. If $\mu = 0$, ν is equivalent to the classical measure boundary trace of u (see Definition 1.1). But if $0 < \mu < C_H(\Omega)$, it can be shown that, for every $\nu \in \mathfrak{M}^+(\partial\Omega)$, the measure boundary trace of $\mathbb{K}^\Omega_\mu[\nu]$ is zero (see Corollary 2.11 below).

In the case $\mu = 0$, the boundary value problem

$$\begin{aligned} -\Delta u + |u|^{q-1} u &= 0 \quad \text{in } \Omega \\ u &= \nu \quad \text{on } \partial\Omega \end{aligned} \tag{1.5}$$

where $q > 1$ and ν is either a finite measure or a positive (possibly unbounded) measure, has been studied by numerous authors. Following Brezis [6], if ν is a finite measure, a weak solution of (1.5) is defined as follows: u is a solution of the problem if u and $\delta|u|^q$ are integrable in Ω and

$$\int_\Omega (-u\Delta\zeta + |u|^{q-1}u\zeta)dx = -\int_{\partial\Omega} \frac{\partial\zeta}{\partial\mathbf{n}} d\nu \quad \forall\zeta \in C_0^2(\overline{\Omega}) \tag{1.6}$$

where $\mathbf{n}$ is the outer unit normal on $\partial\Omega$. Brezis proved that, if a solution exists then it is unique. Gmira and Véron [13] showed that there exists a critical exponent, $q_c := \frac{N+1}{N-1}$, such that if $1 < q < q_c$, (1.6) has a weak solution for every finite measure ν but, if $q \geq q_c$ there exists no positive solution with isolated point singularity.

Marcus and Véron [20] proved that every positive solution of the equation

$$-\Delta u + u^q = 0 \tag{1.7}$$

possesses a boundary trace given by a positive measure ν, not necessarily bounded. In the subcritical case the blow-up set of the trace is a closed set. Furthermore they showed that, in this case, for every such positive measure ν, the boundary value problem (1.5) has a unique solution.

In the case $q = 2$, $N = 2$ this result was previously proved by Le Gall [15] using a probabilistic definition of the boundary trace.

In the supercritical case the problem turned out to be much more challenging. It was studied by several authors using various techniques. The problem was studied by Le Gall, Dynkin, Kuznetsov, Mselati a.o. employing mainly probabilistic methods. Consequently the results applied only to $1 < q \leq 2$. In parallel it was studied by Marcus and Veron employing purely analytic methods that were not subject to the restriction $q \leq 2$. A complete classification of the positive solutions of (1.5) in terms of their behavior at the boundary was provided by Mselati [18] for $q = 2$, by Dynkin [11] for $q_c \leq q \leq 2$ and finally by Marcus [16] for every $q \geq q_c$. For details and related results we refer the reader to [23,22,21,3,10] and the references therein.

In the case of equation (1.1) one is faced by the problem that, according to the classical definition of measure boundary trace, *every positive L_μ-harmonic function has measure boundary trace zero*. Therefore, in order to classify the positive solutions of (1.1) in terms of their behavior at the boundary, it is necessary to introduce a different notion of trace. As in the study of (1.7), we first consider the question of boundary trace for positive L_μ-harmonic or superharmonic functions.

We recall the classical definition of measure boundary trace.

Definition 1.1. (i) A sequence $\{D_n\}$ is a C^2 *exhaustion* of Ω if for every n, D_n is of class C^2, $\overline{D}_n \subset D_{n+1}$ and $\cup_n D_n = \Omega$. If the domains are uniformly of class C^2 we say that $\{D_n\}$ is a uniform C^2 exhaustion.

(ii) Let $u \in W^{1,p}_{loc}(\Omega)$ for some $p > 1$. We say that u possesses a *measure boundary trace* on $\partial\Omega$ if there exists a finite measure ν on $\partial\Omega$ such that, for every uniform C^2 exhaustion $\{D_n\}$ and every $\varphi \in C(\overline{\Omega})$,

$$\lim_{n\to\infty} \int_{\partial D_n} u|_{\partial D_n} \varphi dS = \int_{\partial\Omega} \varphi d\nu.$$

Here $u|_{D_n}$ denotes the Sobolev trace. The measure boundary trace of u is denoted by $\mathrm{tr}\,(u)$.

For $\beta > 0$, denote

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\},\ D_\beta = \{x \in \Omega : \delta(x) > \beta\},\ \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}.$$

Put

$$\alpha_\pm := \frac{1}{2} \pm \sqrt{\frac{1}{4} - \mu}. \tag{1.8}$$

It can be shown (see Corollary 2.11 below) that the classical *measure boundary trace* of $\mathbb{K}^\Omega_\mu[\nu]$ is zero but there exist constants C_1, C_2 such that, for every $\nu \in \mathfrak{M}(\partial\Omega)$,

$$C_1 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \leq \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{K}_\mu^\Omega[\nu](x) dS_x \leq C_2 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \tag{1.9}$$

for all $\beta \in (0, \beta_0)$ where $\beta_0 > 0$ depends only on Ω. In view of this we introduce the following definition of trace.

Definition 1.2. A positive function u possesses a *normalized boundary trace* if there exists a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$\lim_{\beta \to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} |u - \mathbb{K}_\mu^\Omega[\nu]| dS_x = 0. \tag{1.10}$$

The normalized boundary trace will be denoted by $\mathrm{tr}^*(u)$.

Remark. The notion of normalized boundary trace is well defined. Indeed, suppose that ν and ν' satisfy (1.10). Put $v = (\mathbb{K}_\mu^\Omega[\nu - \nu'])_+$ then v is a nonnegative L_μ-subharmonic function, $v \leq \mathbb{K}[\nu + \nu']$ and $\mathrm{tr}^*(v) = 0$. By Proposition 2.14, $v = 0$, i.e., $\mathbb{K}_\mu^\Omega[\nu - \nu'] \leq 0$. By interchanging the roles of ν and ν', we deduce that $\mathbb{K}_\mu^\Omega[\nu' - \nu] \leq 0$. Thus $\nu = \nu'$.

Denote by G_μ^Ω the Green function of $-L_\mu$ in Ω and, for every positive Radon measure τ in Ω, put

$$\mathbb{G}_\mu^\Omega[\tau](x) := \int_\Omega G_\mu^\Omega(x, y) d\tau(y).$$

Denote by $\mathfrak{M}_f(\Omega)$, f a positive Borel function in Ω, the space of Radon measures τ on Ω satisfying $\int_\Omega f\, d|\tau| < \infty$ and by $\mathfrak{M}_f^+(\Omega)$ the positive cone of this space.

If τ is a positive measure such that $\mathbb{G}_\mu^\Omega[\tau](x) < \infty$ for some $x \in \Omega$ then $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ and $\mathbb{G}_\mu^\Omega[\tau]$ is finite everywhere in Ω. The underlying reason for this is the behavior of the Green function at the boundary: for every $\beta > 0$ there exists c_β such that

$$c_\beta^{-1} \delta(x)^{\alpha_+} \leq G_\mu^\Omega(x, y) \leq c_\beta \delta(x)^{\alpha_+} \quad \forall x \in \Omega_{\beta/2},\ y \in D_\beta.$$

For details see Section 2.2 below.

We begin with the study of the linear boundary value problem,

$$\begin{aligned} -L_\mu u &= \tau \quad \text{in } \Omega \\ \mathrm{tr}^*(u) &= \nu, \end{aligned} \tag{1.11}$$

where $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$. As usual we look for solutions $u \in L_{loc}^1(\Omega)$ and the equation is understood in the sense of distributions. The representation theorem implies that if $\tau = 0$ the problem has a unique solution, $u = \mathbb{K}_\mu^\Omega[\nu]$.

We list below our main results regarding this problem.

Proposition I.

(i) *If u is a non-negative L_μ-harmonic function and $\mathrm{tr}^*(u) = 0$ then $u = 0$.*
(ii) *If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$ then $\mathbb{G}_\mu^\Omega[\tau]$ has normalized trace zero. Thus $\mathbb{G}_\mu^\Omega[\tau]$ is a solution of (1.11) with $\nu = 0$.*
(iii) *Let u be a positive L_μ-subharmonic function. If u is dominated by an L_μ-superharmonic function then $L_\mu u \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$ and u has a normalized boundary trace. In this case $\mathrm{tr}^*(u) = 0$ if and only if $u \equiv 0$.*
(iv) *Let u be a positive L_μ-superharmonic function. Then there exist $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$ such that*

$$u = \mathbb{G}_\mu^\Omega[\tau] + \mathbb{K}_\mu^\Omega[\nu]. \tag{1.12}$$

In particular, u is an L_μ-potential (i.e., u does not dominate any positive L_μ-harmonic function) if and only if $\mathrm{tr}^*(u) = 0$.
(v) *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}^+(\Omega)$, problem (1.11) has a unique solution. The solution is given by (1.12).*

Next we study the nonlinear boundary value problem,

$$\begin{aligned} -L_\mu u + u^q &= 0 \quad \text{in } \Omega \\ \operatorname{tr}^*(u) &= \nu \end{aligned} \tag{1.13}$$

where $\nu \in \mathfrak{M}^+(\partial\Omega)$.

Definition 1.3. (i) A positive solution of (1.1) is L_μ*-moderate* if it is dominated by an L_μ-harmonic function.
(ii) A positive function $u \in L^q_{loc}(\Omega)$ is a *(weak) solution of* (1.13) if it satisfies the equation (in the sense of distributions) and has normalized boundary trace ν.

Definition 1.4. Put

$$X(\Omega) = \{\zeta \in C^2(\Omega) : \delta^{\alpha_-} L_\mu \zeta \in L^\infty(\Omega),\ \delta^{-\alpha_+}\zeta \in L^\infty(\Omega)\}.$$

A function $\zeta \in X(\Omega)$ is called an *admissible test function* for (1.13).

Following are our main results concerning the nonlinear problem (1.13). Theorems A–D apply to arbitrary exponent $q > 1$.

Theorem A. *Assume that* $0 < \mu < C_H(\Omega)$, $q > 1$. *Let u be a positive solution of* (1.1). *Then the following statements are equivalent:*
(i) *u is* L_μ*-moderate.*
(ii) *u admits a normalized boundary trace* $\nu \in \mathfrak{M}^+(\partial\Omega)$. *In other words, u is a solution of* (1.13).
(iii) $u \in L^q_{\delta^{\alpha_+}}(\Omega)$ *and*

$$u + \mathbb{G}^\Omega_\mu[u^q] = \mathbb{K}^\Omega_\mu[\nu] \tag{1.14}$$

where $\nu = \operatorname{tr}^*(u)$.
Furthermore, a positive function u is a solution of (1.13) *if and only if* $u/\delta^{\alpha_-} \in L^1(\Omega)$, $\delta^{\alpha_+} u^q \in L^1(\Omega)$ *and*

$$\int_\Omega (-u L_\mu \zeta + u^q \zeta)\,dx = -\int_\Omega \mathbb{K}^\Omega_\mu[\nu] L_\mu \zeta\,dx \quad \forall \zeta \in X(\Omega). \tag{1.15}$$

Theorem B. *Assume* $0 < \mu < C_H(\Omega)$, $q > 1$.

I. UNIQUENESS. *For every* $\nu \in \mathfrak{M}^+(\partial\Omega)$, *there exists at most one positive solution of* (1.13).

II. MONOTONICITY. *Assume* $\nu_i \in \mathfrak{M}^+(\partial\Omega)$, $i = 1, 2$. *Let* u_{ν_i} *be the unique solution of* (1.13) *with* ν *replaced by* ν_i, $i = 1, 2$. *If* $\nu_1 \le \nu_2$ *then* $u_{\nu_1} \le u_{\nu_2}$.

III. A-PRIORI ESTIMATE. *There exists a positive constant* $c = c(N, \mu, \Omega)$ *such that every positive solution u of* (1.13) *satisfies,*

$$\|u\|_{L^1_{\delta^{-\alpha_-}}(\Omega)} + \|u\|_{L^q_{\delta^{\alpha_+}}(\Omega)} \le c\,\|\nu\|_{\mathfrak{M}(\partial\Omega)}. \tag{1.16}$$

Theorem C. *Assume* $0 < \mu < C_H(\Omega)$, $q > 1$. *If* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *and* $\mathbb{K}^\Omega_\mu[\nu] \in L^q_{\delta^{\alpha_+}}(\Omega)$ *then there exists a unique solution of the boundary value problem* (1.13).

Corollary C1. *For every positive function* $f \in L^1(\partial\Omega)$ (1.13) *with* $\nu = f$ *admits a unique positive solution.*

Theorem D. *Assume* $0 < \mu < C_H(\Omega)$, $q > 1$. *If u is a positive solution of* (1.13) *then*

$$\lim_{x \to y} \frac{u(x)}{\mathbb{K}^\Omega_\mu[\nu](x)} = 1 \quad \textit{non-tangentially},\ \nu\textit{-a.e. on } \partial\Omega. \tag{1.17}$$

Let

$$q_{\mu,c} := \frac{N+\alpha_+}{N-1-\alpha_-}. \tag{1.18}$$

In the next two results we show, among other things, that $q_{\mu,c}$ is the *critical exponent* for (1.13). This means that, if $1<q<q_{\mu,c}$ then problem (1.13) has a unique solution for every measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ but, if $q \geq q_{\mu,c}$ then the problem has no solution for some measures ν, e.g. Dirac measure.

In Theorem E we consider the subcritical case $1<q<q_{\mu,c}$ and in Theorem F the supercritical case.

Theorem E. *Assume $0<\mu<C_H(\Omega)$ and $1<q<q_{\mu,c}$. Then:*

I. EXISTENCE AND UNIQUENESS. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$ (1.13) admits a unique positive solution u_ν.*

II. STABILITY. *If $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}^+(\partial\Omega)$ weakly convergent to $\nu \in \mathfrak{M}^+(\partial\Omega)$ then $u_{\nu_n} \to u_\nu$ in $L^1_{\delta^{-\alpha_-}}(\Omega)$ and in $L^q_{\delta^{\alpha_+}}(\Omega)$.*

III. LOCAL BEHAVIOR. *Let $\nu = k\delta_y$, where $k>0$ and δ_y is the Dirac measure concentrated at $y \in \partial\Omega$. Then, under the assumptions of Theorem E, the unique solution of (1.13), denoted by $u_{k\delta_y}$, satisfies*

$$\lim_{x\to y} \frac{u_{k\delta_y}(x)}{K^\Omega_\mu(x,y)} = k. \tag{1.19}$$

Remark. Note that in part III we have 'uniform convergence' not just 'non-tangential convergence' as in Theorem D.

Theorem F. *Assume $0<\mu<C_H(\Omega)$ and $q \geq q_{\mu,c}$. Then for every $k>0$ and $y\in\partial\Omega$, there is no positive solution of (1.1) with normalized boundary trace $k\delta_y$.*

In the first part of the paper we study properties of positive L_μ-harmonic functions and the boundary value problem (1.11). In the second part, these results are applied to a study of the corresponding boundary value problem for the nonlinear equation (1.1). These results yield a complete classification of the positive moderate solutions of (1.1) in the subcritical case. They also provide a framework for the study of positive solutions of (1.1) that may blow up at some parts of the boundary. The existence of such solutions in the subcritical case has been studied (by different methods) in [5]. The boundary trace for positive non-moderate solutions and corresponding boundary value problems will be treated in a forthcoming paper.

The main ingredients used in this paper are: the Representation Theorem previously stated and other basic results of potential theory (see [1]), a sharp estimate of the Green kernel of $-L_\mu$ due to Filippas, Moschini and Tertikas [9], estimates for convolutions in weak L^p spaces (see [23, Section 2.3.2]) and the comparison principle obtained in [4].

2. The linear equation

Throughout this paper we assume that $0<\mu<C_H(\Omega)$.

2.1. *Some potential theoretic results*

We denote by $\mathfrak{M}_{\delta^\alpha}(\Omega)$, $\alpha\in\mathbb{R}$, the space of Radon measures τ on Ω satisfying $\int_\Omega \delta^\alpha(x)d|\tau| < \infty$ and by $\mathfrak{M}^+_{\delta^\alpha}(\Omega)$ the positive cone of $\mathfrak{M}_{\delta^\alpha}(\Omega)$. When $\alpha=0$, we use the notation $\mathfrak{M}(\Omega)$ and $\mathfrak{M}^+(\Omega)$. We also denote by $\mathfrak{M}(\partial\Omega)$ the space of finite Radon measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the positive cone of $\mathfrak{M}(\partial\Omega)$.

Let D be a C^2 domain such that $D \Subset \Omega$ and $h \in L^1(\partial\Omega)$. Denote by $\mathbb{S}_\mu(D,h)$ the solution of the problem

$$\begin{cases} -L_\mu u = 0 & \text{in } D \\ \quad\; u = h & \text{on } \partial D. \end{cases} \tag{2.1}$$

Lemma 2.1. *Let u be L_μ-superharmonic in Ω and D be a C^2 domain such that $D\Subset\Omega$. Then $u \geq \mathbb{S}_\mu(D,u)$ a.e. in D.*

Proof. Since u is L_μ-superharmonic in Ω, there exists $\tau \in \mathfrak{M}^+(\Omega)$ such that $-L_\mu u = \tau$. Let v be the solution of

$$\begin{cases} -L_\mu v = \tau & \text{in } D \\ v = 0 & \text{on } \partial D \end{cases} \tag{2.2}$$

and put $w = \mathbb{S}_\mu(D, u)$. Then $w \geq 0$ and $u|_D = v + w \geq v$. □

Lemma 2.2. *Let u be a nonnegative L_μ-superharmonic and $\{D_n\}$ be a C^2 exhaustion of Ω. Then*

$$\hat{u} := \lim_{n\to\infty} \mathbb{S}_\mu(D_n, u)$$

exists and is the largest L_μ-harmonic function dominated by u.

Proof. By Lemma 2.1, $\mathbb{S}_\mu(D_n, u) \leq u|_{D_n}$, hence the sequence $\{\mathbb{S}_\mu(D_n, u)\}$ is decreasing. Consequently, $\hat{u}$ exists and is an L_μ-harmonic function dominated by u. Next, if v is an L_μ-harmonic function dominated by u then $v \leq \mathbb{S}_\mu(D_n, u)$ for every $n \in \mathbb{N}$. Letting $n \to \infty$ yields $v \leq \hat{u}$. □

Definition 2.3. A nonnegative L_μ-superharmonic function is called an *L_μ-potential* if its largest L_μ-harmonic minorant is zero.

As a consequence of Lemma 2.2, we obtain

Lemma 2.4. *Let u_p be a nonnegative L_μ-superharmonic function in Ω. If for some C^2 exhaustion $\{D_n\}$ of Ω,*

$$\lim_{n\to\infty} \mathbb{S}_\mu(D_n, u_p) = 0, \tag{2.3}$$

then u_p is an L_μ-potential in Ω. Conversely, if u_p is an L_μ-potential, then (2.3) holds for every C^2 exhaustion $\{D_n\}$ of Ω.

For easy reference we quote below the Riesz decomposition theorem (see [1]).

Theorem 2.5. *Every nonnegative L_μ-superharmonic function u in Ω can be written in a unique way in the form $u = u_p + u_h$ where u_p is an L_μ-potential and u_h is a nonnegative L_μ-harmonic function in Ω.*

The next result is a consequence of the Fatou convergence theorem [1, Theorem 1.8] and the following well-known fact: if a function satisfies the local Harnack inequality, fine convergence at the boundary (in the sense of [1]) implies non-tangential convergence.

Theorem 2.6. *Let u_p be a positive L_μ-potential and u be a positive L_μ-harmonic function. Assume that $\frac{u_p}{u}$ satisfies the Harnack inequality. Then*

$$\lim_{x\to y} \frac{u_p(x)}{u(x)} = 0 \quad \textit{non-tangentially, } \nu\textit{-a.e. on } \partial\Omega$$

where ν is the L_μ-boundary measure of u.

2.2. The action of the Green and Martin kernels on spaces of measures

From [2], for every $y \in \partial\Omega$, there exists a positive L_μ-harmonic function in Ω which vanishes on $\partial\Omega \setminus \{y\}$. When normalized, this function is unique. We choose a fixed reference point x_0 in Ω and denote by $K^\Omega_{\mu,y}$ this L_μ-harmonic function, normalized by $K^\Omega_{\mu,y}(x_0) = 1$. The function $K^\Omega_\mu(\cdot, y) = K^\Omega_{\mu,y}(\cdot)$ is the L_μ-Martin kernel in Ω, normalized at x_0.

For $\nu \in \mathfrak{M}(\partial\Omega)$ denote

$$\mathbb{K}^\Omega_\mu[\nu](x) = \int_{\partial\Omega} K^\Omega_\mu(x, y) d\nu(y).$$

In what follows the notation $f \sim g$ means: there exists a positive constant c such that $c^{-1}f < g < cf$ in the domain of the two functions or in a specified subset of this domain. Of course, in the latter case, the constant depends on the subset.

Let G_μ^Ω be the Green kernel for the operator L_μ in $\Omega \times \Omega$. Fix a point $x_0 \in \Omega$. It is well known that the function $x \mapsto G_\mu^\Omega(x, x_0)$ behaves like the first eigenfunction $\varphi_{\mu,1}(x)$ near the boundary, i.e., $G_\mu^\Omega(\cdot, x_0) \sim \varphi_{\mu,1}$ in Ω_β, $(0 < \beta < \delta(x_0))$.

By [19, Lemmas 5,1, 5.2] (see also [8, Lemma 7] for an alternative proof)

$$c^{-1}\delta(x)^{\alpha_+} \le \varphi_{\mu,1}(x) \le c\delta(x)^{\alpha_+}. \tag{2.4}$$

Thus, if $0 < \beta < \delta(x_0)$,

$$c_\beta^{-1}\delta(x)^{\alpha_+} \le G_\mu^\Omega(x, x_0) \le c_\beta\delta(x)^{\alpha_+} \quad \forall x \in \Omega_\beta. \tag{2.5}$$

Therefore, if $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ then

$$\mathbb{G}_\mu^\Omega[\tau](x) := \int_\Omega G_\mu^\Omega(x, y)d\tau(y) < \infty \quad \text{a.e. in } \Omega.$$

Indeed, by (2.5) and the symmetry of the Green kernel, for every $x \in \Omega$, the integral over $\Omega_{\delta(x)/2}$ is finite. For $y \in D_{\delta(x)/4}$, $G_\mu^\Omega(x, y) \le c|x - y|^{2-N}$. Therefore the integral is finite over this set as well. Inequality (2.5) also implies that, if τ is a positive Radon measure in Ω and $\mathbb{G}_\mu^\Omega[\tau](x) < \infty$ for some point $x \in \Omega$ then $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ and $\mathbb{G}_\mu^\Omega[\tau]$ is finite everywhere in Ω.

By [9, Theorem 4.11], for every $x, y \in \Omega$, $x \neq y$,

$$G_\mu^\Omega(x, y) \sim \min\left\{|x - y|^{2-N}, \delta(x)^{\alpha_+}\delta(y)^{\alpha_+}|x - y|^{2\alpha_- - N}\right\} \tag{2.6}$$

Since

$$K_\mu^\Omega(x, y) := \lim_{z \to y} \frac{G_\mu^\Omega(x, z)}{G_\mu^\Omega(x_0, z)} \quad \forall x \in \Omega$$

it follows from (2.6) that

$$K_\mu^\Omega(x, y) \sim \delta(x)^{\alpha_+}|x - y|^{2\alpha_- - N} \quad \forall x \in \Omega, y \in \partial\Omega. \tag{2.7}$$

Let $G^\Omega = G_0^\Omega$ and $P^\Omega = P_0^\Omega$ denote the Green and Poisson kernels of $-\Delta$ in Ω. Then, by (2.7)

$$\frac{K_\mu^\Omega(x, y)}{\delta(x)^{\alpha_-}} \sim \frac{\delta(x)}{|x - y|^N}\left(\frac{|x - y|}{\delta(x)}\right)^{2\alpha_-} \sim P^\Omega(x, y)\left(\frac{|x - y|}{\delta(x)}\right)^{2\alpha_-}. \tag{2.8}$$

Denote $L_w^p(\Omega; \tau)$, $1 \le p < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^p space defined as follows: a measurable function f in Ω belongs to this space if there exists a constant c such that

$$\lambda_f(a; \tau) := \tau(\{x \in \Omega : |f(x)| > a\}) \le ca^{-p}, \quad \forall a > 0. \tag{2.9}$$

The function λ_f is called the distribution function of f (relative to τ). For $p \ge 1$, denote

$$L_w^p(\Omega; \tau) = \{f \text{ Borel measurable} : \sup_{a>0} a^p\lambda_f(a; \tau) < \infty\}$$

and

$$\|f\|^*_{L_w^p(\Omega;\tau)} = (\sup_{a>0} a^p\lambda_f(a; \tau))^{\frac{1}{p}}. \tag{2.10}$$

This expression is not a norm, but for $p > 1$, it is equivalent to the norm

$$\|f\|_{L_w^p(\Omega;\tau)} = \sup\left\{\frac{\int_\omega |f|d\tau}{\tau(\omega)^{1/p'}} : \omega \subset \Omega, \omega \text{ measurable }, 0 < \tau(\omega)\right\}. \tag{2.11}$$

More precisely,

$$\|f\|^*_{L^p_w(\Omega;\tau)} \leq \|f\|_{L^p_w(\Omega;\tau)} \leq \frac{p}{p-1}\|f\|^*_{L^p_w(\Omega;\tau)}. \tag{2.12}$$

Notice that, for every $\alpha > -1$,

$$L^p_w(\Omega;\delta^\alpha dx) \subset L^r_{\delta^\alpha}(\Omega) \quad \forall r \in [1,p).$$

For every $x \in \partial\Omega$, denote by $\mathbf{n}_x$ the outward unit normal vector to $\partial\Omega$ at x.

The following is a well-known geometric property of C^2 domains.

Proposition 2.7. *There exists $\beta_0 > 0$ such that*

(i) For every point $x \in \overline{\Omega}_{\beta_0}$, there exists a unique point $\sigma_x \in \partial\Omega$ such that $|x - \sigma_x| = \delta(x)$. This implies $x = \sigma_x - \delta(x)\mathbf{n}_{\sigma_x}$.

(ii) The mappings $x \mapsto \delta(x)$ and $x \mapsto \sigma_x$ belong to $C^2(\overline{\Omega}_{\beta_0})$ and $C^1(\overline{\Omega}_{\beta_0})$ respectively. Furthermore, $\lim_{x\to\sigma(x)} \nabla\delta(x) = -\mathbf{n}_x$.

By combining (2.6), (2.7) and [23, Lemma 2.3.2], we obtain

Proposition 2.8. *There exist constants c_i depending only on N, μ, β, Ω such that,*

$$\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2}}(\Omega,\delta^\beta)} \leq c_1\|\tau\|_{\mathfrak{M}(\Omega)}, \quad \forall \tau \in \mathfrak{M}(\Omega),\ \beta > -1, \tag{2.13}$$

$$\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2\alpha_-}}(\Omega,\delta^{\beta-\alpha_+})} \leq c_1\|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)}, \quad \forall \tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega),\ \beta > -2\alpha_-, \tag{2.14}$$

$$\left\|\mathbb{K}^\Omega_\mu[\nu]\right\|_{L_w^{\frac{N+\beta}{N-1-\alpha_-}}(\Omega,\delta^\beta)} \leq c_2\|\nu\|_{\mathfrak{M}(\partial\Omega)}, \quad \forall \nu \in \mathfrak{M}(\partial\Omega),\ \beta > -1. \tag{2.15}$$

Proof. We assume that τ is positive; otherwise we replace τ by $|\tau|$. We consider τ as a positive measure in $\mathbb{R}^N$ by extending τ by zero outside of Ω. For $a \in (0,N)$, denote $\Gamma_a(x) = |x|^{a-N}$. By [23, inequality (2.3.17)],

$$\|\Gamma_a * \tau\|_{L_w^{\frac{N+\beta}{N-a}}(\Omega,\delta^\beta)} \leq c\|\tau\|_{\mathfrak{M}(\Omega)} \quad \forall \beta > \max\{-1,-a\} \tag{2.16}$$

where $c = c(N, a, \beta, diam(\Omega))$. By (2.6),

$$G^\Omega_\mu(x,y) \leq c\min\{\Gamma_2(x-y), \delta(x)^{\alpha_+}\delta(y)^{\alpha_+}\Gamma_{2\alpha_-}(x-y)\}.$$

Hence, by (2.16),

$$\begin{aligned}
\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2}}(\Omega,\delta^\beta)} &\leq c\|\Gamma_2 * \tau\|_{L_w^{\frac{N+\beta}{N-2}}(\Omega,\delta^\beta)} \\
&\leq c'\|\tau\|_{\mathfrak{M}(\Omega)} \quad \forall \beta > -1, \\
\left\|\mathbb{G}^\Omega_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N-2\alpha_-}}(\Omega,\delta^{\beta-\alpha_+})} &\leq c\left\|\Gamma_{2\alpha_-} * (\delta^{\alpha_+}\tau)\right\|_{L_w^{\frac{N+\beta}{N-2\alpha_-}}(\Omega,\delta^\beta)} \\
&\leq c\|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)} \quad \forall \beta > -2\alpha_-.
\end{aligned}$$

Next we extend ν by zero outside $\partial\Omega$ and observe that, by (2.7), $K^\Omega_\mu(x,y) \leq c\Gamma_{1+\alpha_-}(x-y)$. Hence $\mathbb{K}^\Omega_\mu[\nu] \leq c\Gamma_{1+\alpha_-} * \nu$ and by (2.16),

$$\left\|\mathbb{K}^\Omega_\mu[\nu]\right\|_{L_w^{\frac{N+\beta}{N-1-\alpha_-}}(\Omega,\delta^\beta)} \leq c\left\|\Gamma_{1+\alpha_-} * \nu\right\|_{L_w^{\frac{N+\beta}{N-1-\alpha_-}}(\Omega,\delta^\beta)} \leq c\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \forall \beta > -1. \qquad \square$$

Corollary 2.9. *Let $\beta > -1$.*

(i) If $\{\nu_n\} \subset \mathfrak{M}^+(\partial\Omega)$ converges weakly to $\nu \in \mathfrak{M}^+(\partial\Omega)$ then $\{\mathbb{K}^\Omega_\mu[\nu_n]\}$ converges to $\mathbb{K}^\Omega_\mu[\nu]$ in $L^p_{\delta^\beta}(\Omega)$ for every p such that $1 \leq p < \frac{N+\beta}{N-1-\alpha_-}$.

(ii) If $\{\tau_n\} \subset \mathfrak{M}^+(\Omega)$ converges weakly (relative to $C_0(\bar{\Omega})$) to $\tau \in \mathfrak{M}^+(\Omega)$ then $\{\mathbb{G}^\Omega_\mu[\tau_n]\}$ converges to $\mathbb{G}^\Omega_\mu[\tau]$ in $L^p_{\delta^\beta}(\Omega)$ for every p such that $1 \leq p < \frac{N+\beta}{N-2}$.

Proof. We prove the first statement. The second is proved in a similar way.

Since $K_\mu^\Omega(x,.) \in C(\partial\Omega)$ for every $x \in \Omega$, $\{\mathbb{K}_\mu^\Omega[\nu_n]\}$ converges to $\mathbb{K}_\mu^\Omega[\nu]$ every where in Ω. By Holder inequality and (2.15), we deduce that $\{(\mathbb{K}_\mu^\Omega[\nu_n])^p\}$ is equi-integrable w.r.t. $\delta^\beta dx$ for any $1 \le p < \frac{N+\beta}{N-1-\alpha_-}$. By Vitali's theorem, $\mathbb{K}_\mu^\Omega[\nu_n] \to \mathbb{K}_\mu^\Omega[\nu]$ in $L^p_{\delta^\beta}(\Omega)$. □

2.3. Estimates related to the normalized trace

Proposition 2.10. *There exist positive constants C_1, C_2 such that, for every $\beta \in (0, \beta_0)$,*

$$C_1\beta^{\alpha_-} \le \int_{\Sigma_\beta} K_\mu^\Omega(x,y)dS_x \le C_2\beta^{\alpha_-} \quad \forall y \in \partial\Omega. \tag{2.17}$$

The constants C_1, C_2 depend on N, Ω, μ but not on y.

Furthermore, for every $r_0 > 0$,

$$\lim_{\beta\to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta \setminus B_{r_0}(y)} K_\mu^\Omega(x,y)dS_x = 0 \quad \forall y \in \partial\Omega. \tag{2.18}$$

For r_0 fixed, the rate of convergence is independent of y.

Proof. By (2.7),

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta \setminus B_{r_0}(y)} K_\mu^\Omega(x,y)dS_x \le c\beta^{\alpha_+-\alpha_-}. \tag{2.19}$$

This implies (2.18).

For the next estimate it is convenient to assume that the coordinates are placed so that $y = 0$ and the tangent hyperplane to $\partial\Omega$ at 0 is $x_N = 0$ with the x_N axis pointing into the domain. For $x \in \mathbb{R}^N$ put $x' = (x_1, \cdots, x_{N-1})$. Pick $r_0 \in (0, \beta_0)$ sufficiently small (depending only on the C^2 characteristic of Ω) so that

$$\frac{1}{2}(|x'|^2 + \delta(x)^2) \le |x|^2 \quad \forall x \in \Omega \cap B_{r_0}(0).$$

Then, if $x \in \Sigma_\beta \cap B_{r_0}(0) =: \Sigma_{\beta,0}$,

$$\frac{1}{4}(|x'| + \beta) \le |x|.$$

This inequality and (2.7) imply,

$$\begin{aligned}\int_{\Sigma_{\beta,0}} K_\mu^\Omega(x,0)dS_x &\le c_0\beta^{\alpha_+} \int_{\Sigma_{\beta,0}} (|x'| + \beta)^{2\alpha_- - N} dS_x \\ &\le c_1\beta^{\alpha_+} \int_{|x'|<r_0} (|x'| + \beta)^{2\alpha_- - N} dx' \\ &\le c_2\beta^{\alpha_+} \int_0^{r_0} (t + \beta)^{2\alpha_- - 2} dt \\ &< c_2\beta^{\alpha_-} \int_1^\infty \tau^{-2\alpha_+} d\tau = \frac{c_2}{2\alpha_+ - 1}\beta^{\alpha_-}.\end{aligned}$$

Thus, for $\beta < r_0$,

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_{\beta,0}} K_\mu^\Omega(x,0)dS_x \le \frac{c_2}{2\alpha_+ - 1}. \tag{2.20}$$

Estimates (2.19) and (2.20) imply the second estimate in (2.17). The first estimate in (2.17) follows from (2.8). □

Since (2.17) holds uniformly w.r. to $y \in \partial\Omega$, an application of Fubini's yields the following.

Corollary 2.11. *For every $\nu \in \mathfrak{M}^+(\partial\Omega)$,*

$$\begin{aligned} C_1 \|\nu\|_{\mathfrak{M}(\partial\Omega)} &\le \liminf_{\beta\to 0} \int_{\Sigma_\beta} \frac{\mathbb{K}_\mu^\Omega[\nu]}{\delta(x)^{\alpha_-}} dS_x \\ &\le \limsup_{\beta\to 0} \int_{\Sigma_\beta} \frac{\mathbb{K}_\mu^\Omega[\nu]}{\delta(x)^{\alpha_-}} dS_x \le C_2 \|\nu\|_{\mathfrak{M}(\partial\Omega)} \end{aligned} \tag{2.21}$$

with C_1, C_2 as in (2.17).

Proposition 2.12. *If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ then*

$$\operatorname{tr}^*(\mathbb{G}_\mu^\Omega[\tau]) = 0 \tag{2.22}$$

and, for $0 < \beta < \beta_0$,

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau] dS_x \le c \int_\Omega \delta^{\alpha_+} d|\tau|, \tag{2.23}$$

where c is a constant depending on μ, Ω.

Proof. We may assume that $\tau > 0$. Denote $v := \mathbb{G}_\mu^\Omega[\tau]$. We start with the proof of (2.23).

By Fubini's theorem and (2.6),

$$\begin{aligned} \int_{\Sigma_\beta} v dS_x \le c\Big(&\int_\Omega \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2-N} dS_x\, d\tau(y) \\ &+ \beta^{\alpha_+} \int_\Omega \int_{\Sigma_\beta \setminus B_{\frac{\beta}{2}}(y)} |x-y|^{2\alpha_- - N} dS_x\, \delta^{\alpha_+}(y) d\tau(y)\Big) = I_1(\beta) + I_2(\beta). \end{aligned}$$

Note that, if $x \in \Sigma_\beta$ and $|x-y| \le \beta/2$ then $\beta/2 \le \delta(y) \le 3\beta/2$. Therefore

$$\begin{aligned} I_1(\beta) &\le c_1 \int_{\Sigma_\beta \cap B_{\frac{\beta}{2}}(y)} |x-y|^{2-\alpha_+ - N} dS_x \int_\Omega \delta(y)^{\alpha_+} d\tau(y) \\ &\le c_1' \int_0^{\beta/2} r^{2-\alpha_+ - N} r^{N-2} dr \int_\Omega \delta(y)^{\alpha_+} d\tau(y) \\ &\le c_1'' \beta^{\alpha_-} \int_\Omega \delta(y)^{\alpha_+} d\tau(y) \end{aligned}$$

and

$$I_2(\beta) \le c_2 \beta^{\alpha_+} \int_{\beta/2}^\infty r^{2\alpha_- - N} r^{N-2} dr \int_\Omega \delta(y)^{\alpha_+} d\tau = c_2' \beta^{\alpha_-} \int_\Omega \delta(y)^{\alpha_+} d\tau.$$

This implies (2.23).

Given $\epsilon \in (0, \|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)})$ and $\beta_1 \in (0, \beta_0)$ put $\tau_1 = \tau \chi_{\bar{D}_{\beta_1}}$ and $\tau_2 = \tau - \tau_1$. Pick $\beta_1 = \beta_1(\epsilon)$ such that

$$\int_{\Omega_{\beta_1}} \delta(y)^{\alpha_+}\, d\tau \leq \epsilon. \tag{2.24}$$

Thus the choice of β_1 depends on the rate at which $\int_{\Omega_\beta} \delta^{\alpha_+}\, d\tau$ tends to zero as $\beta \to 0$.

Put $v_i = \mathbb{G}_\mu^\Omega[\tau_i]$. Then, for $0 < \beta < \beta_1/2$,

$$\int_{\Sigma_\beta} v_1\, dS_x \leq c_3 \beta^{\alpha_+} \beta_1^{2\alpha_- - N} \int_\Omega \delta^{\alpha_+}(y) d\tau_1(y).$$

Thus,

$$\lim_{\beta\to 0} \frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_1\, dS_x = 0. \tag{2.25}$$

On the other hand, by (2.23) and (2.24),

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} v_2\, dS_x \leq c\epsilon \quad \forall \beta < \beta_0. \tag{2.26}$$

This implies that $\mathrm{tr}^*(v) = 0$. $\square$

It is well-known that u is an L_μ-potential if and only if there exists a positive measure τ in Ω such that $u = \mathbb{G}_\mu^\Omega[\tau]$ (see e.g. [1, Theorem 12]). The estimate (2.6) implies that if $\mathbb{G}_\mu^\Omega[\tau] \not\equiv \infty$ then $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. Therefore as a consequence of the previous proposition:

Corollary 2.13. *A positive L_μ-superharmonic function u is a potential if and only if* $\mathrm{tr}^*(u) = 0$.

Remark. Let $D \Subset \Omega$ be a C^2 domain and denote by G_μ^D and P_μ^D the Green and Poisson kernels of L_μ in D. (To avoid misunderstanding we point out that, in the formula defining L_μ, $\delta(x)$ denotes, as before, the distance from x to $\partial\Omega$, not to ∂D.) As every positive L_μ harmonic function has measure boundary trace zero, there is no Poisson kernel for L_μ in Ω. However, L_μ has a Poisson kernel in every C^2 domain D strictly contained in Ω. This follows from the fact that the Green kernel G_μ^D exists and behaves like G_0^D.

Proposition 2.14. *Let w be a non-negative L_μ-subharmonic function. If w is dominated by an L_μ-superharmonic function then $L_\mu w \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega)$ and w has a normalized boundary trace $\nu \in \mathfrak{M}^+(\partial\Omega)$. If, in addition,* $\mathrm{tr}^*(w) = 0$ *then $w = 0$.*

Proof. The first assumption implies that there exists a positive Radon measure λ in Ω such that $-L_\mu w = -\lambda$.

First assume that $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. Then $v := w + \mathbb{G}_\mu^\Omega[\lambda]$ is a non-negative L_μ-harmonic function and consequently, by the representation theorem, $v = \mathbb{K}_\mu^\Omega[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega)$. By Proposition 2.12, $\mathrm{tr}^*(w) = \nu$. If $\nu = 0$ then $v = 0$ and therefore $w = 0$. Now let us drop the assumption on λ.

Let v_β be the unique solution of the boundary value problem,

$$-L_\mu v_\beta = -\lambda_\beta \quad \text{in} \quad D_\beta, \qquad v_\beta = h_\beta \quad \text{on} \quad \partial D_\beta$$

where λ_β is the restriction of λ to D_β and h_β is the restriction of w to ∂D_β. (The uniqueness follows from [4, Lemma 2.3].) The uniqueness implies that $v_\beta = w\lfloor_{D_\beta}$. By assumption there exists a positive L_μ-superharmonic function, say V, such that $w \leq V$. Hence

$$w + \mathbb{G}_\mu^{D_\beta}[\lambda_\beta] = \mathbb{P}_\mu^{D_\beta}[h_\beta] \leq \mathbb{P}_\mu^{D_\beta}[V\lfloor_{\partial D_\beta}] \leq V.$$

This implies that $\mathbb{G}_\mu^\Omega[\lambda] = \lim_{\beta\to 0} \mathbb{G}_\mu^{D_\beta}[\lambda_\beta] < \infty$. For fixed $x \in \Omega$, $G_\mu^\Omega(x, y) \sim \delta(y)^{\alpha_+}$. Therefore the finiteness of $\mathbb{G}_\mu^\Omega[\lambda]$ implies that $\lambda \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. By the first part of the proof w has a normalized trace. $\square$

Remark. See Proposition 2.20 below for a complementary result.

2.4. Test functions

Denote

$$X(\Omega)=\{\zeta\in C^2(\Omega):\delta^{\alpha_-}L_\mu\zeta\in L^\infty(\Omega),\,\delta^{-\alpha_+}\zeta\in L^\infty(\Omega)\}.$$

Proposition 2.15. *For any $\zeta\in X(\Omega)$, $\delta^{\alpha_-}|\nabla\zeta|\in L^\infty(\Omega)$.*

Proof. Let $\zeta\in X(\Omega)$ then there exist a positive constant c_1 and a function $f\in L^\infty(\Omega)$ such that $|\zeta|\le c_1\delta^{\alpha_+}$ and

$$-L_\mu\zeta=\delta^{-\alpha_-}f.$$

Take arbitrary point $x_*\in\Omega_{\beta_0}$ and put $d_*=\frac{1}{2}\delta(x_*)$, $y_*=\frac{1}{d_*}x_*$, $\zeta_*(y)=\zeta(d_*y)$ for $y\in\frac{1}{d_*}\Omega_{d_*}$. Note that if $x\in B_{d_*}(x_*)$ then $y=\frac{1}{d_*}x\in B_1(y_*)$ and $1\le \mathrm{dist}\,(y,\partial(\frac{1}{d_*}\Omega_{d_*}))\le 3$. In $B_1(y_*)$,

$$-\Delta\zeta_*-\frac{\mu}{\mathrm{dist}\,(y,\partial(\frac{1}{d_*}\Omega_{d_*}))^2}\zeta_*=d_*^{2-\alpha_-}\mathrm{dist}\,(y,\partial(\frac{1}{d_*}\Omega_{d_*}))^{-\alpha_-}f(d_*y).$$

By local estimate for elliptic equations [12, Theorem 8.32], there exists a positive constant $c_2=c_2(N,\mu)$ such that

$$\max_{B_{\frac{1}{2}}(y_*)}|\nabla\zeta_*|\le c_2[\max_{B_1(y_*)}|\zeta_*|+\max_{B_1(y_*)}(d_*^{2-\alpha_-}\mathrm{dist}\,(y,\partial(\frac{1}{d_*}\Omega_{d_*}))^{-\alpha_-}|f(d_*y)|].$$

This implies

$$d_*|\nabla\zeta(x_*)|\le c_3(\delta(x_*)^{\alpha_+}+\|f\|_{L^\infty(\Omega)}\delta(x_*)^{2-\alpha_-}),$$

where $c_3=c_3(N,\mu,c_1)$. Therefore

$$|\nabla\zeta(x)|\le c_4\delta(x)^{\alpha_+-1}\quad\forall x\in\Omega_{\beta_0}$$

where $c_4=c_4(N,\mu,c_1,\|f\|_{L^\infty(\Omega)})$. Thus $\delta^{-\alpha_-}|\nabla\zeta|\in L^\infty(\Omega)$. $\square$

Definition 2.16. Let $x_0\in\Omega$ and denote $\tilde\beta(x_0)=\min(\beta_0,\frac{1}{2}\delta(x_0))$. We say that $\tilde G_\mu^\Omega$ is a *proper regularization* of G_μ^Ω relative to x_0 if $\tilde G_\mu^\Omega(x)=G_\mu^\Omega(x_0,x)$ for $x\in\overline{\Omega}_{\tilde\beta(x_0)}$, $\tilde G_\mu^\Omega\in C^2(\Omega)\cap C(\overline{\Omega})$ and $\tilde G_\mu^\Omega\ge 0$ in Ω. Similarly $\tilde\delta$ is a *proper regularization* of δ relative to x_0 if $\tilde\delta(x)=\delta(x)$ for $x\in\overline{\Omega}_{\tilde\beta(x_0)}$, $\tilde\delta\in C^2(\overline{\Omega})$ and $\tilde\delta\ge 0$ in Ω.

Remark. Using (2.6) and (2.4), it is easily verified that the functions $\varphi_{\mu,1}$, $\mathbb{G}_\mu^\Omega[\eta]$ (for $\eta\in L^\infty(\Omega)$), $\tilde G_\mu^\Omega$ and $\tilde\delta^{\alpha_+}$ belong to $X(\Omega)$. Moreover, using Proposition 2.15, one obtains,

$$\zeta\in X(\Omega)\quad\text{and}\quad h\in C^2(\bar\Omega)\Longrightarrow h\zeta\in X(\Omega).$$

In the proofs of the next two propositions we use the following construction. Let $D\Subset\Omega$ be a C^2 domain. The Green function for $-L_\mu$ in D is denoted by G_μ^D. (To avoid misunderstanding we point out that, in the formula defining L_μ, $\delta(x)$ denotes, as before, the distance from x to $\partial\Omega$, not to ∂D.) Given $x_0\in\Omega$ we construct a family of functions $\mathcal{G}(x_0)=\{\tilde G_\mu^{D_\beta}:0<\beta<\frac{1}{2}\tilde\beta(x_0)\}$ such that, for each β, $\tilde G_\mu^{D_\beta}$ is a proper regularization of $G_\mu^{D_\beta}(x_0,\cdot)$ in D_β and $\mathcal{G}(x_0)$ has the following properties:

- For every $\beta\in(0,\frac{1}{2}\tilde\beta(x_0))$, $\tilde G_\mu^{D_\beta}\in C^2(\overline{D}_\beta)$, $\tilde G_\mu^{D_\beta}\ge 0$ and $\tilde G_\mu^{D_\beta}(x)=G_\mu^{D_\beta}(x_0,x)$ for $x\in D_\beta\setminus D_{\tilde\beta(x_0)}$.
- The sequences $\{\tilde G_\mu^{D_\beta}\}$ and $\{L_\mu\tilde G_\mu^{D_\beta}\}$ converge to $\tilde G_\mu^\Omega$ and $L_\mu\tilde G_\mu^\Omega$ respectively, as $\beta\to 0$, a.e. in Ω.
- $\left\|\tilde G_\mu^{D_\beta}+|L_\mu\tilde G_\mu^{D_\beta}|\right\|_{L^\infty(D_\beta)}\le M_{x_0}$ where M_{x_0} is a positive constant independent of β.

$\mathcal{G}(x_0)$ will be called a *uniform regularization* of $\{G_\mu^{D_\beta}\}$.

For any function $h\in C^2(\partial\Omega)$, we say that $\tilde h$ is an *admissible extension* of h relative to x_0 in $\overline{\Omega}$ if $\tilde h(x)=h(\sigma(x))$ for $x\in\Omega_{\tilde\beta(x_0)}$ and $\tilde h\in C^2(\overline{\Omega})$.

2.5. Nonhomogeneous linear equations

Here we discuss the boundary value problem (1.11) in Ω.

Lemma 2.17. *Let $u \in L^1_{loc}(\Omega)$ be a positive solution (in the sense of distributions) of equation*

$$-L_\mu u = \tau \tag{2.27}$$

in Ω where τ is a non-negative Radon measure.

If $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$ then

$$-\int_\Omega \mathbb{G}_\mu^\Omega[\tau] L_\mu \zeta \, dx = \int_\Omega \zeta \, d\tau \quad \forall \zeta \in X(\Omega). \tag{2.28}$$

Proof. We may assume that τ is positive. By Proposition 2.12, $\mathrm{tr}^*(\mathbb{G}_\mu^\Omega[\tau]) = 0$. Therefore, given $\varepsilon > 0$, there exists $\bar\beta = \bar\beta(\varepsilon) < \frac{1}{2}\beta_0$ such that,

$$\frac{1}{\beta^{\alpha_-}} \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau] dS_x < \varepsilon \quad \text{and} \quad \int_{\Omega_\beta} \delta^{\alpha_+} d\tau < \varepsilon \quad \forall \beta \in (0, \bar\beta]. \tag{2.29}$$

Let

$$I(\beta) := \int_{D_\beta} \mathbb{G}_\mu^\Omega[\tau] L_\mu \zeta \, dx + \int_{D_\beta} \zeta \, d\tau.$$

To prove (2.28) we show that

$$\lim_{\beta \to 0} I(\beta) = 0. \tag{2.30}$$

Put

$$\tau_1 := \chi_{\bar D_{\bar\beta}} \tau, \quad \tau_2 := \chi_{\Omega_{\bar\beta}} \tau$$

and, for $0 < \beta < \bar\beta$,

$$I_k(\beta) := \int_{D_\beta} \mathbb{G}_\mu^\Omega[\tau_k] L_\mu \zeta \, dx + \int_{D_\beta} \zeta \, d\tau_k, \quad k = 1, 2.$$

As $|\zeta| \le c\delta^{\alpha_+}$ and $|L_\mu \zeta| \le \frac{c}{\delta^{\alpha_-}}$, (2.29) implies,

$$|I_2(\beta)| \le c\varepsilon \quad \forall \beta \in (0, \bar\beta). \tag{2.31}$$

For every $\beta \in (0, \bar\beta)$,

$$-\int_{D_\beta} \mathbb{G}_\mu^\Omega[\tau_1] L_\mu \zeta \, dx = \int_{D_\beta} \zeta \, d\tau_1 + \int_{\Sigma_\beta} \frac{\partial \mathbb{G}_\mu^\Omega[\tau_1]}{\partial \mathbf{n}} \zeta \, dS_x - \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] \frac{\partial \zeta}{\partial \mathbf{n}} dS_x.$$

Thus

$$I_1(\beta) = -\int_{\Sigma_\beta} \frac{\partial \mathbb{G}_\mu^\Omega[\tau_1]}{\partial \mathbf{n}} \zeta \, dS_x + \int_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] \frac{\partial \zeta}{\partial \mathbf{n}} dS_x =: I_{1,1}(\beta) + I_{1,2}(\beta).$$

By Proposition 2.15 and (2.29),

$$|I_{1,2}(\beta)| \le c\varepsilon \quad \forall \beta \in (0, \bar\beta). \tag{2.32}$$

Next we estimate $I_{1,1}(\beta)$ for $\beta \in (0, \bar{\beta}/2)$. By Fubini,

$$I_{1,1}(\beta) = -\int\limits_{\Sigma_\beta} \frac{\partial}{\partial \mathbf{n}_x} \int\limits_{D_{\bar{\beta}}} G_\mu^\Omega(x,y)d\tau_1(y)\zeta(x)dS_x$$
$$= -\int\limits_{D_{\bar{\beta}}} \int\limits_{\Sigma_\beta} \frac{\partial G_\mu^\Omega(x,y)}{\partial \mathbf{n}_x} \zeta(x)\, dS_x d\tau_1(y).$$

For every $y \in D_{\bar{\beta}}$ the function $x \mapsto G_\mu^\Omega(x,y)$ is L_μ-harmonic in $\Omega_{\bar{\beta}}$. By local elliptic estimates, for every $\xi \in \Sigma_\beta$,

$$\sup_{x \in B_{\beta/4}(\xi)} |\nabla_x G_\mu^\Omega(x,y)| \le c\beta^{-1} \sup_{x \in B_{\beta/2}(\xi)} G_\mu^\Omega(x,y).$$

By Harnack's inequality,

$$\sup_{x \in B_{\beta/2}(\xi)} G_\mu^\Omega(x,y) \le c' \inf_{x \in B_{\beta/2}(\xi)} G_\mu^\Omega(x,y).$$

The constants c, c' are independent of $\beta \in (0, \bar{\beta}/2)$, $y \in D_{\bar{\beta}}$ and $\xi \in \Sigma_\beta$. Therefore we obtain,

$$|\nabla_x G_\mu^\Omega(x,y)| \le C\beta^{-1} G_\mu^\Omega(x,y) \quad \forall x \in \Sigma_\beta,\ \forall y \in D_{\bar{\beta}},\ \forall \beta \in (0, \bar{\beta}/2). \tag{2.33}$$

Hence,

$$|I_{1,1}(\beta)| \le C\beta^{-1} \int\limits_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1]|\zeta|\, dS_x.$$

As $|\zeta(x)| \le c\delta(x)^{\alpha_+}$ it follows that,

$$|I_{1,1}(\beta)| \le C\frac{1}{\beta^{\alpha_-}} \int\limits_{\Sigma_\beta} \mathbb{G}_\mu^\Omega[\tau_1] dS_x.$$

Therefore, by (2.29),

$$|I_{1,1}(\beta)| \le C'\varepsilon \quad \forall \beta \in (0, \bar{\beta}/2). \tag{2.34}$$

Finally (2.30) follows from (2.31), (2.32) and (2.34). □

Theorem 2.18. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. Then:*

(i) Problem (1.11) has a unique solution. The solution is given by

$$u = \mathbb{G}_\mu^\Omega[\tau] + \mathbb{K}_\mu^\Omega[\nu]. \tag{2.35}$$

(ii) There exists a positive constant $c = c(N, \mu, \Omega)$ such that

$$\|u\|_{L^1_{\delta^{-\alpha_-}}(\Omega)} \le c(\|\tau\|_{\mathfrak{M}_{\delta^{\alpha_+}}(\Omega)} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{2.36}$$

(iii) u is a solution of (1.11) if and only if $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and

$$-\int\limits_\Omega u L_\mu \zeta dx = \int\limits_\Omega \zeta d\tau - \int\limits_\Omega \mathbb{K}_\mu^\Omega[\nu] L_\mu \zeta dx \quad \forall \zeta \in X(\Omega). \tag{2.37}$$

Proof. (i) Proposition 2.12 implies that (2.35) is a solution of (1.11).

If u and u' are two solutions of (1.11) then $v := (u - u')_+$ is a nonnegative L_μ-subharmonic function such that $\mathrm{tr}^*(v) = 0$ and $v \le 2\mathbb{G}_\mu^\Omega[|\tau|]$ which is a positive L_μ-superharmonic function. By Proposition 2.14, $v \equiv 0$ and hence $u \le u'$ in Ω. Similarly $u' \le u$, so that $u = u'$.

(ii) In view of (2.14) and (2.15), (2.36) is an immediate consequence of (2.35).

(iii) Let u be the solution of (1.11). By (2.36), $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and by Lemma 2.17 and (2.35), u satisfies (2.37).

Conversely, suppose that $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and satisfies (2.37). We show that u is a solution of (1.11) or, equivalently, of (2.35).

By (2.37) with $\zeta \in C_c^\infty(\Omega)$, u is a solution (in the sense of distributions) of the equation in (1.11). It remains to show that $\operatorname{tr}^*(u) = \nu$. Put $U = u - \mathbb{G}_\mu^\Omega[\tau] - \mathbb{K}_\mu^\Omega[\nu]$ and note that, as $-L_\mu u = \tau$, U is L_μ-harmonic.

Let $z \in \Omega$ and let $\mathcal{G}(z)$ be a uniform regularization of $\{G_\mu^{D_\beta} : 0 < \beta < \frac{1}{2}\tilde{\beta}(z)\}$ (see Section 2.4). Then, for every $\beta \in (0, \frac{1}{2}\tilde{\beta}(z))$, $\tilde{G}_\mu^{D_\beta} \in C_0^2(\overline{D}_\beta)$. Recall that $\tilde{G}_\mu^{D_\beta}(x) = G_\mu^{D_\beta}(z,x)$. Therefore, as $\frac{\partial G_\mu^{D_\beta}(z,x)}{\partial \mathbf{n}_x} = P_\mu^{D_\beta}(z,x)$, $x \in \Sigma_\beta$, we obtain

$$-\int_{D_\beta} U(x) L_\mu \tilde{G}_\mu^{D_\beta}(x)dx = \int_{\Sigma_\beta} U(x) P_\mu^{D_\beta}(z,x) dS_x = U(z). \tag{2.38}$$

The second equality is a consequence of the fact that U is L_μ-harmonic. But $L_\mu \tilde{G}_\mu^{D_\beta}(x) \to L_\mu \tilde{G}_\mu^\Omega(z,x)$ pointwise and the sequence $\{L_\mu \tilde{G}_\mu^{D_\beta}\}$ is bounded by a constant M_z. We observe that $U \in L^1(\Omega)$; in fact by assumption $u \in L^1_{\delta^{-\alpha_-}}(\Omega)$ and therefore, by Proposition 2.8, $U \in L^1_{\delta^{-\alpha_-}}(\Omega)$. Consequently, by (2.38),

$$U(z) = -\int_\Omega U(x) L_\mu \tilde{G}_\mu^\Omega(z,x)dx.$$

Since $G_\mu^\Omega(z,\cdot) \in X(\Omega)$, by (2.37) the right hand side vanishes. Thus U vanishes in Ω, i.e., u satisfies (2.35). □

Corollary 2.19. *Let u be a positive L_μ superharmonic function. Then there exist $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega)$ such that (1.12) holds.*

Proof. By the Riesz decomposition theorem u can be written in the form $u = u_p + u_h$ where u_p is an L_μ-potential and u_h is a non-negative L_μ-harmonic function. Therefore there exists $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $u_h = \mathbb{K}_\mu^\Omega[\nu]$. Since u_p is an L_μ-potential there exists a positive Radon measure τ such that $u_p = \mathbb{G}_\mu^\Omega[\tau]$ (see e.g. [1, Theorem 12]). This necessarily implies that $\tau \in \mathfrak{M}_{\delta^{\alpha_+}}(\Omega)$. □

Proposition 2.20. *Let w be a non-negative L_μ-subharmonic function. If w has a normalized boundary trace then it is dominated by an L_μ-harmonic function.*

Proof. There exist a positive Radon measure τ in Ω and a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$-L_\mu w = -\tau \quad \text{in } \Omega, \quad \operatorname{tr}^*(w) = \nu.$$

Let u_β be the solution of

$$-L_\mu u = -\tau_\beta \quad \text{in } D_\beta, \quad u = \mathbb{K}_\mu^\Omega[\nu] \quad \text{on } \Sigma_\beta$$

where $\tau_\beta := \tau\chi_{D_\beta}$. Then,

$$u_\beta + \mathbb{G}_\mu^{D_\beta}[\tau_\beta] = \mathbb{K}_\mu^\Omega[\nu].$$

Letting $\beta \to 0$ we obtain,

$$\mathbb{G}_\mu^\Omega[\tau] \le \mathbb{K}_\mu^\Omega[\nu].$$

Hence $\tau \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\Omega)$ and consequently

$$w + \mathbb{G}_\mu^\Omega[\tau] = \mathbb{K}_\mu^\Omega[\nu]. \qquad \square \tag{2.39}$$

3. The nonlinear equation

In this section, we consider the nonlinear equation

$$-L_\mu u + u^q = 0 \tag{3.1}$$

in Ω with $0 < \mu < C_H(\Omega)$ and $q > 1$.

Proof of Theorem A. Since u is a positive solution of (1.1), u is L_μ-subharmonic. Assuming (i), u is dominated by an L_μ-harmonic function. Therefore, by Proposition 2.14, (i) $\Longrightarrow$ (ii) and $u \in L^q_{\delta^{\alpha_+}}(\Omega)$. On the other hand, by Proposition 2.20 (ii) $\Longrightarrow$ (i).

As mentioned above, (i) implies that $u \in L^q_{\delta^{\alpha_+}}(\Omega)$ and that there exists $\nu \in \mathfrak{M}^+_{\delta^{\alpha_+}}(\partial\Omega)$ such that $\operatorname{tr}^*(u) = \nu$. Therefore, by Theorem 2.18, (1.14) is a consequence of (2.37). Thus (i) $\Longrightarrow$ (iii).

Finally, the implication (iii) $\Longrightarrow$ (i) is obvious.

It remains to prove the last assertion. If u is a positive solution of (1.13) then, by (iii), $u \in L^q_{\delta^{\alpha_+}}(\Omega)$ and (1.15) follows from Theorem 2.18.

Conversely, assume that $\delta^{\alpha_+}u^q$, $u/\delta^{\alpha_-} \in L^1(\Omega)$ and (1.15) holds. Then, by (1.15) with $\zeta \in C_c^\infty(\Omega)$, u is a solution of (1.1). Taking $\zeta_f = \mathbb{G}_\mu^\Omega[f]$ where $f \in C_c(\Omega)$ and $f \geq 0$ we obtain

$$\int_\Omega (\mathbb{K}_\mu^\Omega[\nu] - u) f \, dx = \int_\Omega u^q \zeta_f \, dx < \infty.$$

This implies $u \leq \mathbb{K}_\mu^\Omega[\nu]$, i.e., u is L_μ-moderate. Therefore by (i), u is a solution of (1.13). □

Proof of Theorem B.
Uniqueness. Let u_1 and u_2 be two positive solutions of (1.13). Then $v := (u_1 - u_2)_+$ is a subsolution of (1.1) and therefore an L_μ-subharmonic function. Furthermore, by (iii) in Theorem A, $u_1, u_2 \in L^q_{\delta^{\alpha_+}}(\Omega)$ and $v \leq \mathbb{G}_\mu^\Omega[u_1^q + u_2^q] =: \bar{v}$. Obviously $\bar{v}$ is L_μ superharmonic and $\operatorname{tr}^*(v) = 0$. Therefore, by Proposition 2.14, $v = 0$. Thus $u_1 \leq u_2$ and similarly $u_2 \leq u_1$.

Monotonicity. As before, $v := (u_1 - u_2)_+$ is L_μ-subharmonic and it is dominated by an L_μ-superharmonic function. Since $\nu_1 \leq \nu_2$, $\operatorname{tr}^*(v) = 0$. Hence by Proposition 2.14, $v = 0$.

A-priori estimate. Suppose that u is a positive solution of (1.13). Then (1.15) with $\zeta = \mathbb{G}_\mu^\Omega[1]$ implies (1.16). (Recall that $\mathbb{G}_\mu^\Omega[1] \sim \delta^{\alpha_+}$.) □

For the proof of the next theorem we need

Lemma 3.1. *Let $D \Subset \Omega$ be a C^2 domain and $q > 1$. If h is a positive function in $L^1(\partial D)$ then there exists a unique solution of the boundary value problem,*

$$\begin{aligned} -L_\mu u + u^q &= 0 \quad \text{in } D \\ u &= h \quad \text{on } \partial D. \end{aligned} \tag{3.2}$$

Proof. First assume that h is bounded. Let P_μ^D denote the Poisson kernel of $-L_\mu$ in D and put $u_0 := \mathbb{P}_\mu^D[h]$. Thus u_0 is bounded. We show that there exists a non-increasing sequence of positive functions $\{u_n\}_1^\infty$, dominated by u_0, such that u_n is the solution of the boundary value problem,

$$\begin{aligned} -\Delta v + v^q &= \frac{\mu}{\delta^2} u_{n-1} \quad \text{in } D \\ v &= h \quad \text{on } \partial D \quad n = 1, 2, \ldots \end{aligned} \tag{3.3}$$

As usual δ denotes the distance to $\partial\Omega$, not to ∂D. For $n = 1$, u_0 is a supersolution of the problem and, obviously $v = 0$ is a subsolution. Consequently there exists a unique solution u_1. By induction, for $n > 1$,

$$-\Delta u_{n-1}+u_{n-1}^q=\frac{\mu}{\delta^2}u_{n-2}\geq\frac{\mu}{\delta^2}u_{n-1}.$$

Thus $v=u_{n-1}$ is a supersolution of (3.3) and it is bounded. It follows that there exists $0\leq u_n\leq u_{n-1}$ such that

$$-\Delta u_n+u_n^q=\frac{\mu}{\delta^2}u_{n-1}\text{ in }D,\quad u_n=h\text{ on }\partial D.$$

As the sequence is monotone we conclude that $u=\lim u_n$ is a solution of (3.2).

If $h\in L^1(\partial D)$, we approximate it by a monotone increasing sequence of non-negative bounded functions $\{h_k\}$. If v_k is the solution of (3.2) with h replaced by h_k then $\{v_k\}$ increases (by the comparison principle [4, Lemma 3.2]) and $v=\lim v_k$ is a solution of (3.2).

Uniqueness follows by the comparison principle. □

Proof of Theorem C. Put $u_0:=\mathbb{K}_\mu^\Omega[\nu]$ and $h_\beta:=u_0\lfloor_{\Sigma_\beta}$. Let u_β be the solution of (3.2) with h replaced by h_β, $\beta\in(0,\beta_0)$. Since u_0 is a supersolution of (1.1) it follows that $\{u_\beta\}$ decreases as $\beta\downarrow 0$. Therefore $u:=\lim_{\beta\to 0}u_\beta$ is a solution of (1.1).

We claim that $\operatorname{tr}^*(u)=\nu$. Indeed,

$$u_\beta+\mathbb{G}_\mu^{D_\beta}[u_\beta^q]=\mathbb{P}_\mu^{D_\beta}[h_\beta]=u_0. \tag{3.4}$$

Furthermore, in D_β, $u_\beta\leq u_0\in L^q_{\delta^{\alpha_+}}(\Omega)$. Therefore

$$\mathbb{G}_\mu^{D_\beta}[u_\beta^q]\to\mathbb{G}_\mu^\Omega[u^q].$$

Hence, by (3.4),

$$u+\mathbb{G}_\mu^\Omega[u^q]=u_0=\mathbb{K}_\mu^\Omega[\nu].$$

By Proposition 2.12, $\operatorname{tr}^*(u)=\nu$.

By Theorem B the solution is unique. □

Proof of Corollary C1. By the previous theorem, if $\nu=f$ where f is a positive bounded function then (1.13) has a solution. If $0\leq f\in L^1(\Omega)$ then it is the limit of an increasing sequence of such functions. Therefore, once again problem (1.13) with $\nu=f$ has a solution.

Proof of Theorem D. Put $v=\mathbb{K}_\mu^\Omega[\nu]-u$. By the comparison principle $v\geq 0$. Clearly v is L_μ-superharmonic in Ω and, by definition $\operatorname{tr}^*(v)=0$. By Proposition I(iv) v is an L_μ potential. Consequently, by Theorem 2.6,

$$\lim_{x\to y}\frac{v(x)}{\mathbb{K}_\mu^\Omega[\nu]}=0\quad\text{non-tangentially, }\nu\text{ a.e. on }\partial\Omega.$$

This implies (1.17). □

Proof of Theorem E. By Proposition 2.8, specifically inequality (2.15), $\mathbb{K}_\mu^\Omega[\nu]\in L^q_{\delta^{\alpha_+}}(\Omega)$ for every $q\in(1,q_{\mu,c})$ and $\nu\in\mathfrak{M}^+(\partial\Omega)$. Therefore the first assertion of the theorem is a consequence of Theorem C.

We turn to the proof of stability. Put $v_n=\mathbb{K}_\mu^\Omega[\nu_n]$. By Proposition 2.8, $\{v_n\}$ is bounded in $L^q_{\delta^{\alpha_+}}(\Omega)$ for every $q\in(1,q_{\mu,c})$ and in $L^p_{\delta^{-\alpha_-}}(\Omega)$ for every $p\in(1,\frac{N-\alpha_-}{N-1-\alpha_-})$. In addition $v_n\to v$ pointwise in Ω. This implies that $\{v_n^q\delta^{\alpha_+}\}$ and $\{v_n/\delta^{\alpha_-}\}$ are uniformly integrable in Ω. Since $u_{\nu_n}\leq v_n$ it follows that this conclusion applies also to $\{u_{\nu_n}\}$.

By the extension of the Keller–Osserman inequality due to [4], the sequence $\{u_{\nu_n}\}$ is uniformly bounded in every compact subset of Ω. Therefore, by a standard argument, we can extract a subsequence, still denoted by $\{u_{\nu_n}\}$ that converges pointwise to a solution u of (1.1). In view of the uniform convergence mentioned above we conclude that

$$u_{\nu_n}\to u\quad\text{in }L^q_{\delta^{\alpha_+}}(\Omega)\text{ and in }L^1_{\delta^{-\alpha_-}}(\Omega).$$

By Theorem A,

$$u_{\nu_n}+\mathbb{G}_\mu^\Omega[u_{\nu_n}^q]=\mathbb{K}_\mu^\Omega[\nu_n].$$

In view of the previous observations, passing to the limit as $n \to \infty$, we obtain,

$$u + \mathbb{G}_\mu^\Omega[u^q] = \mathbb{K}_\mu^\Omega[\nu].$$

Again by Theorem A it follows that u is the (unique) solution of (1.13). Because of the uniqueness we conclude that the entire sequence $\{u_{\nu_n}\}$ (not just a subsequence) converges to u as stated in assertion II. of the theorem.

Finally we prove assertion III. By Theorem A

$$u_{k\delta_y} + \mathbb{G}_\mu^\Omega[u_{k\delta_y}^q] = kK_\mu^\Omega(\cdot, y). \tag{3.5}$$

Combining (2.7), (2.6) and the fact $u_{k\delta_y} \le kK_\mu^\Omega(\cdot, y)$, we obtain

$$\frac{\mathbb{G}_\mu^\Omega[u_{k\delta_y}^q](x)}{K_\mu^\Omega(x, y)} \le k^q \frac{\mathbb{G}_\mu^\Omega[(K_\mu^\Omega(., y)^q](x)}{K_\mu^\Omega(x, y)} \le ck^q|x-y|^{N+\alpha_+-q(N-1-\alpha_-)}.$$

Since $1 < q < q_{\mu,c}$, it follows that

$$\lim_{x\to y} \frac{\mathbb{G}_\mu^\Omega[u_{k\delta_y}^q](x)}{K_\mu^\Omega(x, y)} = 0.$$

Therefore, by (3.5), we obtain (1.19). $\square$

Proof of Theorem F. Let $y \in \partial\Omega$. By negation, assume that there exists a positive solution u of (1.13) with $\nu = k\delta_y$ for some $k > 0$. By Theorem A, $u \le k\mathbb{K}_\mu^\Omega(., y)$ and $u \in L_{\delta^{\alpha_+}}^q(\Omega)$. Let $\gamma \in (0, 1)$ and denote $C_\gamma(y) = \{x \in \Omega : \gamma|x-y| \le \delta(x)\}$. By Theorem D,

$$\lim_{x\in C_\gamma(y), x\to y} \frac{u(x)}{K_\mu^\Omega(x, y)} = k.$$

This implies that there exist positive numbers r_0, c such that

$$u(x) \ge cK_\mu^\Omega(x, y) \quad \forall x \in C_\gamma(y) \cap B_{r_0}(y). \tag{3.6}$$

By (2.7),

$$\begin{aligned} J_\gamma := & \int_{C_\gamma(y)\cap B_{r_0}(y)} (K_\mu^\Omega(x, y))^q \delta(x)^{\alpha_+} dx \\ & \ge c' \int_{C_\gamma(y)\cap B_{r_0}(y)} \delta(x)^{\alpha_+(q+1)}|x-y|^{(2\alpha_- - N)q} dx \\ & \ge c'\gamma^{\alpha_+(q+1)} \int_{C_\gamma(y)\cap B_{r_0}(y)} |x-y|^{\alpha_+ - q(N-1-\alpha_-)} dx. \end{aligned}$$

Since $q \ge q_{\mu,c}$ the last integral is divergent. But (3.6) and the fact that $u \in L_{\delta^{\alpha_+}}^q(\Omega)$ imply that $J_\gamma < \infty$. We reached a contradiction. $\square$

Conflict of interest statement

No conflict of interest.

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References

[1] A. Ancona, Theorié du potentiel sur les graphes et les variétés, in: Ecole d'été de Probabilités de Saint-Flour XVIII-1988, in: Springer Lecture Notes in Math., vol. 1427, 1990, pp. 1–112.

[2] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, Ann. Math. (2) 125 (1987) 495–536.

[3] A. Ancona, M. Marcus, Positive solutions of a class of semilinear equations with absorption and Schrödinger equations, J. Math. Pures Appl. 104 (2015) 587–618.

[4] C. Bandle, V. Moroz, W. Reichel, Boundary blowup type sub-solutions to semilinear elliptic equations with Hardy potential, J. Lond. Math. Soc. 2 (2008) 503–523.

[5] C. Bandle, M. Marcus, V. Moroz, Boundary singularities of solutions to elliptic equations with Hardy potential, preprint.

[6] H. Brezis, Une équation semilinéaire avec conditions aux limites dans L^1, unpublished note, 1972.

[7] H. Brezis, M. Marcus, Hardy inequalities revisited, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (4) 25 (1997) 217–237.

[8] J. Dávila, L. Dupaigne, Hardy-type inequalities, J. Eur. Math. Soc. 6 (2004) 335–365.

[9] S. Filippas, L. Moschini, A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrodinger operators on bounded domains, Commun. Math. Phys. 273 (2007) 237–281.

[10] E.B. Dynkin, Diffusions, Superdiffusions and Partial Differential Equations, Amer. Math. Soc., Providence, RI, 2002.

[11] E.B. Dynkin, Superdiffusions and Positive Solutions of Nonlinear Partial Differential Equations, Amer. Math. Soc., Providence, RI, 2004.

[12] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, second edition, Springer, Berlin, 1983.

[13] A. Gmira, L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations, Duke Math. J. 64 (1991) 271–324.

[14] J.B. Keller, On solutions of $\Delta u = f(u)$, Commun. Pure Appl. Math. 10 (1957) 503–510.

[15] J.F. Le Gall, The Brownian snake and solutions of $\Delta u = u^2$ in a domain, Probab. Theory Relat. Fields 102 (1995) 393–432.

[16] M. Marcus, Complete classification of the positive solutions of $-\Delta u + u^q = 0$, J. Anal. Math. 117 (2012) 187–220.

[17] M. Marcus, V.J. Mizel, Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^N$, Trans. Am. Math. Soc. 350 (1998) 3237–3255.

[18] B. Mselati, Classification and probabilistic representation of the positive solutions of a semilinear elliptic equations, Mem. Am. Math. Soc. 168 (798) (2004).

[19] M. Marcus, I. Shafrir, An eigenvalue problem related to Hardy's L^p inequality, Ann. Sc. Norm. Super. Pisa, Cl. Sci. 29 (2000) 581–604.

[20] M. Marcus, L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, Arch. Ration. Mech. Anal. 144 (1998) 201–231.

[21] M. Marcus, L. Véron, Removable singularities and boundary trace, J. Math. Pures Appl. 80 (2001) 879–900.

[22] M. Marcus, L. Véron, Capacitary estimates of positive solutions of semilinear elliptic equations with absorbtion, J. Eur. Math. Soc. 6 (2004) 483–527.

[23] M. Marcus, L. Véron, Nonlinear Second Order Elliptic Equations Involving Measures, De Gruyter Series in Nonlinear Analysis and Applications, 2013.

[24] R. Osserman, On the inequality $\Delta u \geq f(u)$, Pac. J. Math. 7 (1957) 1641–1647.

CHAPTER 3

Semilinear elliptic equations with a Hardy potential and a subcritical source term

This chapter is based on the paper [**119**]. In this chapter, we discuss semilinear elliptic equations with a source term and a Hardy potential. Various necessary and sufficient conditions for the existence of solutions to the corresponding Dirichlet problem are obtained.

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Semilinear elliptic equations with Hardy potential and subcritical source term

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Abstract Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N > 2)$ and $\delta(x) := \operatorname{dist}(x, \partial\Omega)$. Assume $\mu \in \mathbb{R}_+$, ν is a nonnegative finite measure on $\partial\Omega$ and $g \in C(\Omega \times \mathbb{R}_+)$. We study positive solutions of

$$-\Delta u - \frac{\mu}{\delta^2} u = g(x, u) \text{ in } \Omega, \qquad \operatorname{tr}^*(u) = \nu. \tag{P}$$

Here $\operatorname{tr}^*(u)$ denotes the *normalized boundary trace* of u which was recently introduced by Marcus and Nguyen (Ann Inst H Poincaré Anal Non Linéaire, 34, 69–88, 2017). We focus on the case $0 < \mu < C_H(\Omega)$ (the Hardy constant for Ω) and provide qualitative properties of positive solutions of (P). When $g(x, u) = u^q$ with $q > 0$, we prove that there is a critical value q^* (depending only on N, μ) for (P) in the sense that if $q < q^*$ then (P) possesses a solution under a smallness assumption on ν, but if $q \geq q^*$ this problem admits no solution with isolated boundary singularity. Existence result is then extended to a more general setting where g is *subcritical* [see (1.28)]. We also investigate the case where g is linear or sublinear and give an existence result for (P).

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1 Introduction

This paper concerns a study of weak solutions of semilinear elliptic equations with Hardy potential and source term

$$- \Delta u - \frac{\mu}{\delta^2} u = g(x,u) \quad \text{in } \Omega \tag{1.1}$$

where Ω is a C^2 bounded domain in $\mathbb{R}^N (N > 2)$, $\mu \geq 0$, $\delta(x) := \operatorname{dist}(x, \partial\Omega)$ and $g \in C(\Omega \times \mathbb{R}_+)$.

Henceforth, we will use the notations $L_\mu := \Delta + \frac{\mu}{\delta^2}$ and $(g \circ u)(x) := g(x, u(x))$.

Definition 1.1 (i) A function u is an L_μ-harmonic function (resp. L_μ-subharmonic, L_μ-superharmonic) in Ω if $u \in L^1_{\text{loc}}(\Omega)$ and

$$-L_\mu u = 0 \text{ (resp.} - L_\mu u \leq 0, \ -L_\mu u \geq 0)$$

in the sense of distributions in Ω.

(ii) A function u is called a nonnegative weak solution (resp. subsolution, supersolution) of (1.1) if $u \geq 0$, $u \in L^1_{\text{loc}}(\Omega)$, $g \circ u \in L^1_{\text{loc}}(\Omega)$ and

$$-L_\mu u = g \circ u \text{ (resp.} - L_\mu u \leq g \circ u, -L_\mu u \geq g \circ u)$$

in the sense of distributions in Ω.

Boundary value problem with measures for (1.1) with $\mu = 0$ and $g \circ u = u^q$, i.e. the problem

$$-\Delta u = u^q \quad \text{in } \Omega, \qquad u = \nu \quad \text{on } \partial\Omega \tag{1.2}$$

was first considered by Bidault-Véron and Vivier [7]. They established estimates involving classical Green and Poisson kernels for $-\Delta$ and applied these estimates to obtain an existence result in the subcritical case, i.e. $1 < q < q_c := \frac{N+1}{N-1}$. Then Bidaut-Véron and Yarur [9] reconsidered this type of problem in a more general setting and provided a necessary and sufficient condition for the existence of a solution of (1.2). Chen et al. [12] investigated (1.1) with $\mu = 0$ and g satisfying a *subcriticality condition*. Their approach makes use of Schauder fixed point theorem, essentially based on estimates related to weighted Marcinkiewicz spaces. Recently, Bidaut-Véron et al. [8] provided new criteria for the existence of weak solutions of problem (1.2) and extended those results to the case where Δ is replaced by L_μ.

When $\mu \neq 0$, the study of (1.1) relies strongly on the investigation of the linear equation

$$- L_\mu u = 0 \quad \text{in } \Omega. \tag{1.3}$$

Equation (1.3) with $\mu < 0$, and more generally Schrödinger equations $-\Delta u + V(x)u = 0$ where V is a nonnegative potential, was studied by Ancona [1,2], Marcus [18], Ancona and Marcus [3] and by Véron and Yarur [25]. The case $\mu > 0$ was considered by Bandle et al. [4–6], Marcus and Nguyen [20], Gkikas and Véron [15] and by Marcus and Moroz [19] in connection with the Hardy constant $C_H(\Omega)$ which is given by

$$C_H(\Omega) = \inf_{H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\int_\Omega (u/\delta)^2 dx}. \tag{1.4}$$

It is well known (see [11,21]) that $C_H(\Omega) \in (0, \frac{1}{4}]$ and $C_H(\Omega) = \frac{1}{4}$ when Ω is convex. Moreover the infimum is achieved if and only if $C_H(\Omega) < 1/4$.

Let $\phi \geq 0$ in Ω and $p \geq 1$, we denote by $L^p(\Omega; \phi)$ the space of all functions v on Ω satisfying $\int_\Omega |v|\,\phi\, dx < \infty$. We denote by $\mathfrak{M}(\Omega; \phi)$ the space of Radon measures τ on Ω satisfying $\int_\Omega \phi\, d|\tau| < \infty$ and by $\mathfrak{M}^+(\Omega; \phi)$ the nonnegative cone of $\mathfrak{M}(\Omega; \phi)$. When $\phi \equiv 1$, we use the usual notations $\mathfrak{M}(\Omega)$ and $\mathfrak{M}^+(\Omega)$. We also denote by $\mathfrak{M}(\partial\Omega)$ the space of finite measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the nonnegative cone of $\mathfrak{M}(\partial\Omega)$.

Let G_μ and K_μ be the Green and the Martin kernels for $-L_\mu$ in Ω respectively (see [20] for more detail). Denote by $\mathbb{G}_\mu$ and $\mathbb{K}_\mu$ the associated operators defined by

$$\mathbb{G}_\mu[\tau](x) = \int_\Omega G_\mu(x, y) d\tau(y), \quad \forall \tau \in \mathfrak{M}(\Omega), \tag{1.5}$$

$$\mathbb{K}_\mu[\nu](x) = \int_{\partial\Omega} K_\mu(x, z) d\nu(z), \quad \forall \nu \in \mathfrak{M}(\partial\Omega). \tag{1.6}$$

Put

$$\alpha_\pm := \frac{1 \pm \sqrt{1 - 4\mu}}{2}. \tag{1.7}$$

Let $\lambda_{\mu,1}$ be the first eigenvalue of $-L_\mu$ in Ω and denote by $\varphi_{\mu,1}$ the corresponding eigenfunction normalized by $\int_\Omega (\varphi_{\mu,1}/\delta)^2 dx = 1$ (see [11]). If $\mu \in (0, C_H(\Omega))$ then $\lambda_{\mu,1} > 0$ and by [13] (see also [22]), there exists a constant $c_1 > 0$ such that

$$c_1^{-1} \delta^{\alpha_+} \leq \varphi_{\mu,1} \leq c_1 \delta^{\alpha_+} \quad \text{in } \Omega. \tag{1.8}$$

For $\beta > 0$, put

$$\Omega_\beta = \{x \in \Omega : \delta(x) < \beta\}, \; D_\beta = \{x \in \Omega : \delta(x) > \beta\}, \; \Sigma_\beta = \{x \in \Omega : \delta(x) = \beta\}.$$

When dealing with boundary value problem associated to (1.1) with $\mu > 0$ one encounters the following difficulties:

- The first one is due to the fact that every positive L_μ-harmonic function has classical measure boundary trace zero (see [20, Corollary 2.11]). Therefore, the classical notion of boundary trace no longer plays an important role in describing the boundary behavior of L_μ-harmonic function or solutions of (1.1).
- The second one stems from the invalidity of the classical Keller-Osserman estimate, as well as the lack of a universal upper bound for solutions of (1.1). Moreover, contrast to the case of nonnegative absorption nonlinearity, $\mathbb{K}_\mu[\nu]$ with $\nu \in \mathfrak{M}^+(\partial\Omega)$ is a subsolution of

$$-L_\mu u = g \circ u \quad \text{in } \Omega \tag{1.9}$$

and therefore it is no longer a natural upper bound for solutions of (1.9).

In order to overcome the first difficulty, we shall employ the notion of *normalized boundary trace* which is defined as follows:

Definition 1.2 A function u possesses a *normalized boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that

$$\lim_{\beta\to 0} \beta^{-\alpha_-} \int_{\Sigma_\beta} |u(x) - \mathbb{K}_\mu[\nu](x)| dS(x) = 0. \tag{1.10}$$

The normalized boundary trace of u is denoted by $\mathrm{tr}^*(u)$.

In the above definition, we use the notation $dS = d\mathbb{H}_{N-1}$ where $\mathbb{H}_{N-1}$ denotes the Hausdorff measure. This notion was introduced by Marcus and Nguyen [20] in the case $\mu \in (0, C_H(\Omega))$. It is worth mentioning that $\lambda_{\mu,1} > 0$ when $\mu \in (0, C_H(\Omega))$ and hence $\varphi_{\mu,1}$ is a positive L_μ-superharmonic function in Ω. This fact, together with a classical result of Ancona [2], guarantees the validity of Representation theorem (see [20]). The notion of normalized boundary trace turned out to be appropriate to investigate the problem

$$-L_\mu u + u^q = 0 \quad \text{in } \Omega, \qquad \mathrm{tr}^*(u) = \nu. \tag{1.11}$$

More precisely, when $\mu \in (0, C_H(\Omega))$, they showed that there exists a critical exponent

$$q^* = q^*(N, \mu) := \frac{N + \alpha_+}{N + \alpha_+ - 2}. \tag{1.12}$$

for (1.11). This means that if $1 < q < q^*$, for every positive finite boundary measure ν on $\partial\Omega$, (1.11) admits a unique positive solution, while if $q \geq q^*$ there exists no positive solution of (1.11) with ν being a Dirac measure. Stability result was also discussed in the case $1 < q < q^*$. Problem (1.11) with u^q replaced by a more general nonlinearity $f(u)$ was then investigated by Gkikas and Véron [15] in a slightly different setting. When $f(u) = |u|^{q-1}u$, they provided a necessary and sufficient condition in terms of Besov capacity for solving (1.11) in the supercritical case, i.e. $q \geq q^*$.

Because of the second difficulty, we mainly deal with the minimal solution of (1.9) which possesses several exploitable properties. This solution is constructed due to sub-supersolutions theorem in Sect. 3. Observe that $\mathbb{K}_\mu[\nu]$ with $\nu \in \mathfrak{M}^+(\partial\Omega)$ is a subsolution of (1.9); hence in order to prove the existence of a minimal solution of (1.9), it is sufficient to find a supersolution of (1.9) which dominates $\mathbb{K}_\mu[\nu]$.

Throughout the present paper, we assume that $\mu \in (0, C_H(\Omega))$. We now introduce the definition of solutions of

$$-L_\mu u = g \circ u \quad \text{in } \Omega, \quad \mathrm{tr}^*(u) = \nu. \tag{1.13}$$

Definition 1.3 (i) A nonnegative function u is called a *(weak) solution of* (1.13) if u is a solution of (1.1) and has normalized boundary trace ν.

(ii) Put

$$X(\Omega) := \{\zeta \in C^2(\Omega) : \delta^{\alpha_-} L_\mu \zeta \in L^\infty(\Omega),\ \delta^{-\alpha_+}\zeta \in L^\infty(\Omega)\}.$$

A function $\zeta \in X(\Omega)$ is called an *admissible test function* for (1.13).

Notice that $\varphi_{\mu,1} \in X(\Omega)$. More properties of $X(\Omega)$ can be found in [20, Section 2.4]. Using this space, we establish integral formulation for weak solutions of (1.13). This is stated in the following result.

Theorem A *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$. The following statements are equivalent:*

(i) *u is a positive weak solution of* (1.13)*;*
(ii) $g \circ u \in L^1(\Omega; \delta^{\alpha_+})$ *and*

$$u = \mathbb{G}_\mu[g \circ u] + \mathbb{K}_\mu[\nu]; \tag{1.14}$$

(iii) $u \in L^1(\Omega; \delta^{-\alpha_-})$, $g \circ u \in L^1(\Omega; \delta^{\alpha_+})$ *and*

$$-\int_\Omega u L_\mu \zeta dx = \int_\Omega (g \circ u)\zeta dx - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta dx \quad \forall \zeta \in X(\Omega). \tag{1.15}$$

Under some additional assumptions on g, we obtain an existence result for (1.13).

Theorem B *Let $g(x, r)$ be a nondecreasing continuous function with respect to r for every $x \in \Omega$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. Assume that there exist numbers $c_2 > 0$, $c_3 > 0$, $0 \le r_1 < r_2 \le \infty$ and a function $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$g(x, rs) \le \ell(r) g(x, s) \quad \forall s \ge 0, r > 0, x \in \Omega, \tag{1.16}$$

$$\ell(1 + c_2 c_3 r^{-1} \ell(r)) \le c_2 \quad \forall r \in (r_1, r_2), \tag{1.17}$$

$$\mathbb{G}_\mu[g \circ (\mathbb{K}_\mu[\nu])] \le c_3 \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega. \tag{1.18}$$

1. EXISTENCE. *For any $\varrho \in (r_1, r_2)$ the problem*

$$-L_\mu u = g \circ u \quad \text{in } \Omega, \qquad \operatorname{tr}^*(u) = \varrho\nu \tag{1.19}$$

admits a minimal positive weak solution $\underline{u}_{\varrho\nu}$ in the sense that if v is a positive weak solution of (1.19) *then $\underline{u}_{\varrho\nu} \le v$ in Ω.*

2. ESTIMATES. There exists a positive constant $c_4 = c_4(c_2, c_3, \ell, \varrho)$ such that

$$\varrho \mathbb{K}_\mu[\nu] \le \underline{u}_{\varrho\nu} \le c_4 \varrho \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega. \tag{1.20}$$

3. NONTANGENTIAL CONVERGENCE. For ν-a.e. point $z \in \partial\Omega$, there holds

$$\lim_{x \to z} \frac{\underline{u}_{\varrho\nu}(x)}{\mathbb{K}_\mu[\nu](x)} = \varrho \quad \text{non-tangentially.} \tag{1.21}$$

Remark When $g(x, u) = u^q$ with $q > 1$, c_4 can be chosen independently of ϱ.

In the next results, we focus on the pure power case, namely the problem

$$-L_\mu u = u^q \quad \text{in } \Omega, \qquad \operatorname{tr}^*(u) = \nu \tag{D_ν}$$

where $q > 0$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. We shall establish some estimates related Green and Martin operators and a necessary condition for the existence of solutions of (D_ν) in the case $q > 1$.

Theorem C *Let $q > 0$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Then there exists a positive constant $c_5 = c_5(N, \mu, q, \Omega)$ such that*

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q] \le c_5 \|\nu\|^{q-1}_{\mathfrak{M}(\partial\Omega)} \mathbb{K}_\mu[\nu] \quad \text{a.e in } \Omega. \tag{1.22}$$

Furthermore, if $q > 1$ and problem (D_ν) admits a positive weak solution then there holds

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q] \le \frac{1}{q-1} \mathbb{K}_\mu[\nu] \quad \text{a.e. in } \Omega. \tag{1.23}$$

Remark It is worth mentioning that when $\mu = 0$, (1.22) and (1.23) were obtained by Bidaut-Véron and Vivier [7]. When L_μ is replaced by a uniformly elliptic differential operator of second order with bounded Hölder-continuous coefficients, (1.23) is relevant to [17, Theorem 7.6]. Recently, (1.23) with an inexplicit multiplier on the right hand-side were proved by Véron et al. in [8, Theorem 4.1]. In this paper we employ the method in [7] to prove (1.22) for $q > 0$ and apply the idea in [7,9,10] to point out that the multiplier on the right hand-side of (1.23) can be explicitly chosen as $\frac{1}{q-1}$. When $q = 1$, estimate (1.22) becomes $\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]] \le c_5 \mathbb{K}_\mu[\nu]$ with $c_5 = c_5(N, \mu, \Omega)$, which can be regarded as the limiting case of (1.23).

The next results reveal that q^* is a *critical exponent* for (D_ν). More precisely, in the subcritical case, namely $1 < q < q^*$, (D_ν) admits a solution under a smallness assumption on the boundary datum, while in the supercritical case, i.e. $q \ge q^*$, this problem possesses no solution with isolated boundary singularity.

For $z \in \partial\Omega$, we denote by δ_z the Dirac measure concentrated at z. Existence and nonexistence results when $0 < q < q^*$, $q \ne 1$ are given as follows.

Theorem D *Let $q \in (0, q^*)$, $q \ne 1$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. For $\varrho > 0$, consider the problem*

$$-L_\mu u = u^q \quad in\ \Omega, \qquad \mathrm{tr}^*(u) = \varrho\nu. \tag{$D_{\varrho\nu}$}$$

1. CASE: $q \in (1, q^*)$. There is a threshold value $\varrho^* \in \mathbb{R}_+$ for $(D_{\varrho\nu})$ such that the following holds.

(i) If $\varrho \in (0, \varrho^*]$ then problem $(D_{\varrho\nu})$ admits a minimal positive weak solution $\underline{u}_{\varrho\nu}$. Moreover, if $\varrho \in (0, \varrho^*)$, $\underline{u}_{\varrho\nu}$ satisfies (1.20) and (1.21). In addition, if $\{\varrho_n\}$ is a nondecreasing sequence converging to ϱ^* then $\{\underline{u}_{\varrho_n\nu}\}$ converges to $\underline{u}_{\varrho^*\nu}$ in $L^1(\Omega; \delta^{-\alpha_-})$ and in $L^q(\Omega; \delta^{\alpha_+})$.

(ii) If $\varrho > \varrho^*$ then there exists no positive weak solution of $(D_{\varrho\nu})$.

2. CASE: $q \in (0, 1)$. For every $\varrho > 0$ problem $(D_{\varrho\nu})$ admits a minimal solution $\underline{u}_{\varrho\nu}$ which satisfies satisfies (1.20) and (1.21). Moreover, $\lim_{\varrho\to\infty} \underline{u}_{\varrho\nu} = \infty$ a.e. in Ω.

For any $1 \ne q \in (0, q^*)$, if $\nu = \delta_z$ with $z \in \partial\Omega$ then there holds

$$\lim_{x\to z} \frac{\underline{u}_{\varrho\delta_z}(x)}{K_\mu(x, z)} = \varrho. \tag{1.24}$$

Remark Note that in the absorption case [namely equation (1.11)], if $1 < q < q^*$, there are two types of solution with isolated boundary singularity: the weakly singular solutions $u_{\varrho,z}$ (the solution of (1.11) with $\nu = \varrho\delta_z$) and the strongly singular solution $u_{\infty,z}$. Actually, $u_{\infty,z}$ is the limit of the sequence $u_{\varrho,z}$ as $\varrho \to \infty$. This limiting process can not be executed in the source case since $(D_{\varrho\delta_z})$ admits no solution if $\varrho > \varrho^*$ due to Theorem D.

We next give a stability result.

Theorem E *Let $q \in (0, q^*)$, $q \ne 1$ and $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}^+(\partial\Omega)$ which converges weakly to $\nu \in \mathfrak{M}^+(\partial\Omega)$. If $q > 1$, assume in addition that*

$$\sup_n \|\nu_n\|_{\mathfrak{M}(\partial\Omega)} \le \varrho^*. \tag{1.25}$$

For each n, let u_{ν_n} be a positive weak solution of (D_{ν_n}). Then, up to a subsequence, $\{u_{\nu_n}\}$ converges to a positive weak solution u_ν of (D_ν) in $L^1(\Omega; \delta^{-\alpha_-})$ and in $L^q(\Omega; \delta^{\alpha_+})$.

An existence and stability result in the case $q = 1$ is stated in the following theorem in which $\lambda_{\mu,1}$ is the first eigenvalue of $-L_\mu$ in Ω.

Theorem F *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$. For $\kappa > 0$, consider the problem*

$$-L_\mu u = \kappa u \quad \text{in } \Omega, \qquad \mathrm{tr}^*(u) = \nu. \tag{E_ν^κ}$$

There exists a number $\kappa^* \in (0, \lambda_{\mu,1}]$ such that the following holds.

(i) *If $\kappa \in (0, \kappa^*)$ then problem (E_ν^κ) admits a minimal positive weak solution $\underline{u}_{\kappa,\nu}$. Moreover, $\underline{u}_{\kappa,\nu}$ satisfies (1.21).*
Assume $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}^+(\partial\Omega)$ which converges weakly to $\nu \in \mathfrak{M}^+(\partial\Omega)$ and for each n denote by u_{κ,ν_n} a positive weak solution of $(E_{\nu_n}^\kappa)$. Then, up to a subsequence, $\{u_{\kappa,\nu_n}\}$ converges to a positive weak solution $u_{\kappa,\nu}$ of (E_ν^κ) in $L^1(\Omega; \delta^{-\alpha_-})$.

(ii) *If $\kappa > \kappa^*$ then (E_ν^κ) admits no positive weak solution.*

Furthermore, problem $(E_\nu^{\lambda_{\mu,1}})$ admits no positive weak solution.

Remark It is notified by the referee that $\kappa^* = \lambda_{\mu,1}$. The way to prove it is to note that $-L_\mu - \kappa$ admits the Green function $G_{\mu,\kappa}$ for any $\kappa < \mu$ and then to prove a modification of Proposition 2.4 for $G_{\mu,\kappa}$ (see [24] for the existence of the Green function $G_{\mu,\kappa}$). The weaker statement $\kappa^* \le \lambda_{\mu,1}$ in the present paper is essentially in order to simplify the proofs and to streamline the exposition.

In the supercritical case, i.e. $q \ge q^*$, there is no solution with an isolated boundary singularity.

Theorem G *Assume $q \ge q^*$. Then for every $\varrho > 0$ and $z \in \partial\Omega$, there is no positive weak solution of*

$$-L_\mu u = u^q \quad \text{in } \Omega, \qquad \mathrm{tr}^*(u) = \varrho\delta_z. \tag{$D_{\varrho\delta_z}$}$$

Here δ_z denotes the Dirac measure concentrated at z.

Remark Interesting removability results for (D_ν) in terms of capacities in the supercritical case was provided in [8].

In the next two theorems, we consider the case $(g \circ u)(x) = \delta(x)^\gamma \tilde{g}(u(x))$ where $\gamma > -1 - \alpha_+$ and $\tilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and continuous. In this framework, the critical exponent for (1.1) is

$$q_\gamma^* = q_\gamma^*(N, \mu, \gamma) := \frac{N + \alpha_+ + \gamma}{N + \alpha_+ - 2}. \tag{1.26}$$

Clearly $q_0^* = q^*$.

Theorem H gives an existence result for the problem

$$-L_\mu u = \delta^\gamma \tilde{g}(u) \quad \text{in } \Omega, \qquad \mathrm{tr}^*(u) = \varrho\nu. \tag{1.27}$$

Theorem H *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. Assume that*

$$\Lambda_0 := \int_1^\infty s^{-1-q_\gamma^*} \tilde{g}(s)\,ds < +\infty, \tag{1.28}$$

$$\tilde{g}(s) \le \Lambda_1 s^{q_1} + \theta, \quad \forall s \in [0,1] \quad \text{for some } q_1 > 1,\ \Lambda_1 > 0,\ \theta > 0. \tag{1.29}$$

Then there exist $\theta_0 > 0$ and $\varrho_0 > 0$ depending on $N, \mu, \gamma, \Lambda_0, \Lambda_1$ and q_1 such that for every $\theta \in (0, \theta_0)$ and $\varrho \in (0, \varrho_0)$ problem (1.27) admits a weak solution $u \ge \varrho\mathbb{K}_\mu[\nu]$ in Ω.

Remark If $\tilde{g}$ satisfies (1.28) we say that $\tilde{g}$ is *subcritical with respect to* γ.

The case where $\tilde{g}$ is *linear* or *sublinear* is treated in the following theorem.

Theorem I *Let* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *with* $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. *Assume that*

$$\tilde{g}(s) \leq \Lambda_2 s^{q_2} + \theta, \quad \forall s \geq 0 \tag{1.30}$$

for some $q_2 \in (0, 1]$, $\Lambda_2 > 0$ *and* $\theta > 0$.

In (1.30), *if* $q_2 = 1$ *we assume in addition that* Λ_2 *is small enough. Then for any* $\varrho > 0$, (1.27) *admits a weak solution* $u \geq \varrho\mathbb{K}_\mu[\nu]$.

Remark In Theorem I, when $q_2 < 1$, the smallness assumption on θ is not required.

The plan of the paper is as follows. In Sect. 2 we give results concerning Green and Martin kernels and boundary value problem for linear equations with Hardy potential. Theorems A and B are proved in Sect. 3. It is noteworthy that main ingredients in proving Theorem A are: a generalization of Herglotz–Doob theorem to L_μ-superharmonic functions and the theory of Schrödinger linear equations. Theorem B is established using a sub-supersolutions theorem. The proof of Theorems C–G are presented in Sect. 4. Finally, in Sect. 5 the existence result in the case of more general source terms (Theorems H and I) is obtained due to the Schauder fixed point theorem and estimates in weak L^p spaces.

2 Preliminaries

Throughout this paper we assume that $0 < \mu < C_H(\Omega)$.

2.1 Weak L^p spaces

We denote by $L^p_w(\Omega; \tau)$, $1 \leq p < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^p space (or Marcinkiewicz space) (see [23]). When $\tau = \delta^\alpha dx$, for simplicity, we use the notation $L^p_w(\Omega; \delta^\alpha)$. Notice that, for every $\alpha > -1$,

$$L^p_w(\Omega; \delta^\alpha dx) \subset L^r(\Omega; \delta^\alpha), \qquad \forall r \in [1, p).$$

If $u \in L^p_w(\Omega; \delta^\alpha)(\alpha > -1)$ then

$$\int_{|\{u| \geq s\}} \delta^\alpha dx \leq s^{-p} \|u\|^p_{L^p_w(\Omega;\delta^\alpha)}. \tag{2.1}$$

2.2 Green and Martin kernels

Let G_μ be the Green kernel for the operator $-L_\mu$ in $\Omega \times \Omega$ and denote by $\mathbb{G}_\mu$ the associated operator defined by (1.5). It was shown in [20] that for every $\tau \in \mathfrak{M}(\Omega; \delta^{\alpha_+})$, $|\mathbb{G}_\mu[\tau]| < \infty$ a.e. in Ω. Denote by K_μ the Martin kernel for $-L_\mu$ in Ω and by $\mathbb{K}_\mu$ the Martin operator defined by (1.6).

In what follows the notation $f \sim g$ means: there is a constant $c > 0$ such that $c^{-1} f < g < cf$ in the domain of the two functions.

By [14, Theorem 4.11] and [20] (see also [15]),

$$G_\mu(x, y) \sim \min\left\{|x - y|^{2-N}, \delta(x)^{\alpha_+}\delta(y)^{\alpha_+} |x - y|^{2\alpha_- - N}\right\} \quad \forall x, y \in \Omega,\ x \neq y; \tag{2.2}$$

$$K_\mu(x, z) \sim \delta(x)^{\alpha_+}|x - z|^{2\alpha_- - N} \quad \forall x \in \Omega, z \in \partial\Omega. \tag{2.3}$$

The following estimates can be found in [16, Proposition 2.4]

Proposition 2.1 (i) *Let* $\beta \in \left(-\frac{N\alpha_+}{N+2\alpha_+-2}, \frac{N\alpha_+}{N-2}\right)$. *Then there exists a constant* $c_7 = c_7(N,\mu,\beta,\Omega)$ *such that*

$$\left\|\mathbb{G}_\mu[\tau]\right\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega;\delta^\beta)} \leq c_7 \,\|\tau\|_{\mathfrak{M}(\Omega;\delta^{\alpha_+})} \quad \forall \tau \in \mathfrak{M}(\Omega;\delta^{\alpha_+}). \tag{2.4}$$

(ii) *Let* $\beta > -1$. *Then there exists a constant* $c_8 = c_8(N,\mu,\beta,\Omega)$ *such that*

$$\left\|\mathbb{K}_\mu[\nu]\right\|_{L_w^{\frac{N+\beta}{N+\alpha_+-2}}(\Omega;\delta^\beta)} \leq c_8 \,\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \forall \nu \in \mathfrak{M}(\partial\Omega). \tag{2.5}$$

2.3 Some results on linear equations

In this subsection, we recall some results concerning boundary value problem for non homogeneous linear equation

$$-L_\mu u = \tau \quad \text{in } \Omega. \tag{2.6}$$

Definition 2.2 (i) A function u is a solution of (2.6) if $u \in L^1_{loc}(\Omega)$ and (2.6) is understood in the sense of distributions.

(ii) Let $\tau \in \mathfrak{M}(\Omega;\delta^{\alpha_+})$ and $\nu \in \mathfrak{M}(\partial\Omega)$. A function u is a weak solution of

$$-L_\mu u = \tau \quad \text{in } \Omega, \qquad \operatorname{tr}^*(u) = \nu, \tag{2.7}$$

if u is a solution of (2.6) and u admits normalized boundary trace ν.

Definition 2.3 A nonnegative L_μ-superharmonic function is called an L_μ-potential if its largest L_μ-harmonic minorant is zero.

The following results, which can be found in [20, Proposition I], is crucial in proving Theorem A.

Proposition 2.4 (i) *If* $\tau = 0$ *then problem* (2.7) *has a unique weak solution* $u = \mathbb{K}_\mu[\nu]$. *If* u *is a nonnegative* L_μ*-harmonic function and* $\operatorname{tr}^*(u) = 0$ *then* $u = 0$.

(ii) *If* $\tau \in \mathfrak{M}^+(\Omega;\delta^{\alpha_+})$ *then* $\operatorname{tr}^*(\mathbb{G}_\mu[\tau]) = 0$. *Thus* $\mathbb{G}_\mu[\tau]$ *is a solution of* (2.7) *with* $\nu = 0$.

(iii) *Let* u *be a positive* L_μ*-subharmonic function. If* u *is dominated by an* L_μ*-superharmonic function then* $L_\mu u \in \mathfrak{M}^+(\Omega;\delta^{\alpha_+})$ *and* u *has a normalized boundary trace. In this case* $\operatorname{tr}^*(u) = 0$ *if and only if* $u \equiv 0$.

(iv) *Let* u *be a positive* L_μ*-superharmonic function. Then there exist* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *and* $\tau \in \mathfrak{M}^+(\Omega;\delta^{\alpha_+})$ *such that*

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu]. \tag{2.8}$$

In particular, u *is an* L_μ*-potential if and only if* $\operatorname{tr}^*(u) = 0$.

(v) *For every* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *and* $\tau \in \mathfrak{M}^+(\Omega;\delta^{\alpha_+})$, (2.7) *has a unique positive solution which is given by* (2.8). *Moreover, there exists a positive constant* $c_9 = c_9(N,\mu,\Omega)$ *such that*

$$\|u\|_{L^1(\Omega;\delta^{-\alpha_-})} \leq c_9(\|\tau\|_{\mathfrak{M}(\Omega;\delta^{\alpha_+})} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{2.9}$$

(vi) *u is a solution of of* (2.7) *if and only if* $u \in L^1(\Omega;\delta^{-\alpha_-})$ *and*

$$-\int_\Omega u L_\mu \zeta \, dx = \int_\Omega \zeta \, d\tau - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta \, dx, \quad \forall \zeta \in X(\Omega). \tag{2.10}$$

For easy reference, we present a potential theoretic result which serves to prove Theorem B.

Theorem 2.5 *Let w_1 be a positive L_μ-potential and w_2 be a positive L_μ-harmonic function with $\nu = \operatorname{tr}^*(w_2)$. Assume that $\frac{w_1}{w_2}$ satisfies the local Harnack inequality. Then for ν-a.e. $z \in \partial\Omega$,*

$$\lim_{x \to z} \frac{w_1(x)}{w_2(x)} = 0 \quad \textit{non-tangentially.}$$

This theorem can be obtained by combining the Fatou convergence theorem [1, Theorem 1.8] and the fact that if a function satisfies the Harnack inequality, fine convergence at the boundary (in the sense of [1]) implies non-tangential convergence (for more details, see [3]).

3 Nonlinear equations with source term

In this section, we deal with nonlinear equations involving source term

$$-L_\mu u = g \circ u \tag{3.1}$$

in Ω where $0 < \mu < C_H(\Omega)$ and $g : \Omega \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous.

3.1 Properties of weak solutions

For $z \in \partial\Omega$, denote by $\mathbf{n}_z$ the outward unit normal vector to $\partial\Omega$ at z. We recall below a geometric property of C^2 domains (see [23]).

Proposition 3.1 *There exists $\beta_0 > 0$ such that for every point $x \in \overline{\Omega}_{\beta_0}$, there exists a unique point $\sigma_x \in \partial\Omega$ such that $x = \sigma_x - \delta(x)\mathbf{n}_{\sigma_x}$. The mappings $x \mapsto \delta(x)$ and $x \mapsto \sigma_x$ belong to $C^2(\overline{\Omega}_{\beta_0})$ and $C^1(\overline{\Omega}_{\beta_0})$ respectively. Moreover, $\lim_{x \to \sigma(x)} \nabla\delta(x) = -\mathbf{n}_{\sigma_x}$.*

For $D \Subset \Omega$, let G_μ^D and K_μ^D be the Green and Poisson kernels of $-L_\mu$ in D respectively. Denote by $\mathbb{G}_\mu^D$ and $\mathbb{K}_\mu^D$ the corresponding Green and Poisson operators in D.

We prove below main properties of solutions of (1.13).

Proof of Theorem A **(i) $\Longrightarrow$ (ii).** Assume u is a positive weak solution of (1.13). Put $\tau = g \circ u$ and for $\beta \in (0, \beta_0)$ denote $\tau_\beta := \tau|_{D_\beta}$ and $\lambda_\beta := u|_{\Sigma_\beta}$. Consider the boundary value problem

$$-L_\mu v = \tau_\beta \quad \text{in } D_\beta, \qquad v = \lambda_\beta \quad \text{on } \Sigma_\beta.$$

This problem admits a unique solution v_β (the uniqueness is derived from [5, Lemma 2.1] since $\mu < C_H(\Omega)$). Therefore $v_\beta = u|_{D_\beta}$. We have

$$u|_{D_\beta} = v_\beta = \mathbb{G}_\mu^{D_\beta}[\tau_\beta] + \mathbb{K}_\mu^{D_\beta}[\lambda_\beta].$$

It follows that

$$\int_{D_\beta} G_\mu^{D_\beta}(\cdot, y)(g \circ u)(y)\, dy = \mathbb{G}_\mu^{D_\beta}[\tau_\beta] \le u|_{D_\beta}.$$

Letting $\beta \to 0$, we get

$$\int_\Omega G_\mu(\cdot, y)(g \circ u)(y)\, dy \le u. \tag{3.2}$$

Fix a point $x_0 \in \Omega$ such that $u(x_0) < \infty$. Keeping in mind that $G_\mu(x_0, y) > c_{x_0}\delta(y)^{\alpha_+}$ for every $y \in \Omega$, we deduce from (3.2) that $g \circ u \in L^1(\Omega; \delta^{\alpha_+})$. Thanks to Proposition 2.4 (v), we obtain (1.14).

(ii) $\Longrightarrow$ (i). Assume u is a function such that $g \circ u \in L^1(\Omega; \delta^{\alpha_+})$ and (1.14) holds. By Proposition 2.4 (i) $-L_\mu \mathbb{K}_\mu[\nu] = 0$, which implies that u is a solution of (3.1). On the other hand, since $g \circ u \in L^1(\Omega; \delta^{\alpha_+})$, we deduce from Proposition 2.4 (ii) that $\operatorname{tr}^*(\mathbb{G}_\mu[g \circ u]) = 0$. Consequently, $\operatorname{tr}^*(u) = \operatorname{tr}^*(\mathbb{K}_\mu[\nu]) = \nu$.

(i) $\Longrightarrow$ (iii). Assume u is a positive solution of (1.13). From the implication (i) $\Longrightarrow$ (ii), we deduce that $u \in L^1(\Omega; \delta^{-\alpha_-})$ and $g \circ u \in L^1(\Omega; \delta^{\alpha_+})$. Hence, by Proposition 2.4 (vi), u satisfies (1.15).

(iii) $\Longrightarrow$ (i). This implication follows directly from Proposition 2.4 (vi). □

3.2 Nondecreasing source

We start with an existence result for (3.1) in presence of sub and super solutions.

Theorem 3.2 *Let $g \in C(\Omega \times \mathbb{R}_+)$, $g(x, r)$ be nondecreasing with respect to r for any $x \in \Omega$. Assume that there exist a subsolution V_1 and a supersolution V_2 of (3.1) such that $0 \le V_1 \le V_2$ in Ω. Then there exists a solution u of (3.1) which satisfies $V_1 \le u \le V_2$ in Ω.*

Moreover, if $V_1 = \mathbb{K}_\mu[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $g \circ V_2 \in L^1(\Omega; \delta^{\alpha_+})$ then there exists a minimal positive weak solution $\underline{u}_\nu$ of (1.13) in the sense that $\underline{u}_\nu \le v$ in Ω for every positive weak solution v of (1.13).

Lemma 3.3 *Let $D \Subset \Omega$, $f \in L^1(D)$, $f \ge 0$ and $\eta \in L^1(\partial D)$, $\eta \ge 0$. Then there exists a unique solution of*

$$-L_\mu u = f \quad \text{in } D, \qquad u = \eta \quad \text{on } \partial D. \tag{3.3}$$

Proof We start with the case $f \in L^2(D)$ and $\eta = 0$. Let us consider the functional

$$\mathcal{J}(v) := \frac{1}{2}\int_D \left(|\nabla v|^2 - \frac{\mu}{\delta^2}v^2\right) dx - \int_D f v dx$$

over the space $H_0^1(D)$. Since $\mu < C_H(\Omega)$, by Hardy inequality and the variational method, one can show that the problem $\min_{H_0^1(D)} \mathcal{J}(v)$ admits a solution $v \in H_0^1(D)$. The minimizer v is the unique weak solution of (3.3).

If $f \in L^1_+(D)$ then we can approximate it by an increasing sequence $\{f_m\} \subset L^\infty_+(D)$. Let v_m be the solution of (3.3) with $\eta = 0$ and f replaced by f_m. By comparison principle [5, Lemma 2.1], $\{v_m\}$ increases and therefore $v := \lim_{m\to\infty} v_m$ is a solution of (3.3) with $\eta = 0$.

We next consider the case $\eta \in L^1(\partial D)$. Let v be a solution of (3.3) with $\eta = 0$ then $u = v + \mathbb{P}^D_\mu[\eta]$ is a solution of (3.3). The uniqueness follows from the comparison principle. □

Proof of Theorem 3.2 Put $u_0 := V_1$ and $\eta_\beta := V_1|_{\Sigma_\beta}$ for $\beta \in (0, \beta_0)$. For $n \ge 1$, consider the problem

$$-L_\mu u = g \circ u_{n-1} \quad \text{in } D_\beta, \qquad u = \eta_\beta \quad \text{on } \partial D_\beta. \tag{3.4}$$

For each $n \ge 1$, by Lemma 3.3 there exists a unique solution $u_{\beta,n}$ of (3.4). Moreover, since $g(x, r)$ is nondecreasing with respect to r for every $x \in \Omega$, by applying the comparison principle, we deduce that

$$V_1 \le u_{\beta,n} \le u_{\beta,n+1} \le V_2$$

in D_β. Therefore $u_\beta := \lim_{n\to\infty} u_{\beta,n}$ is a solution of (3.1) in D_β which satisfies $V_1 \le u_\beta \le V_2$ in D_β. Moreover,

$$u_\beta = \mathbb{G}_\mu^{D_\beta}[g \circ u_\beta] + \mathbb{P}_\mu^{D_\beta}[\eta_\beta]. \tag{3.5}$$

For $0 < \beta' < \beta < \beta_0$, by the comparison principle, $u_{\beta,1} \le u_{\beta',1}$ in D_β. By the monotonicity assumption on g, it follows that $u_{\beta,n} \le u_{\beta',n}$ in D_β for every $n > 1$. Therefore $V_1 \le u_\beta \le u_{\beta'} \le V_2$ in D_β and hence $u := \lim_{\beta\downarrow 0} u_\beta$ is a solution of (3.1) in Ω satisfying $V_1 \le u \le V_2$.

In the case $V_1 = \mathbb{K}_\mu[\nu]$, formulation (3.5) becomes

$$u_\beta = \mathbb{G}_\mu^{D_\beta}[g \circ u_\beta] + \mathbb{K}_\mu[\nu]. \tag{3.6}$$

Put $\underline{u}_\nu := \lim_{\beta\downarrow 0} u_\beta$. Since $0 \le g \circ u_\beta \le g \circ V_2 \in L^1(\Omega; \delta^{\alpha_+})$, it follows that

$$\lim_{\beta\downarrow 0} \mathbb{G}_\mu^{D_\beta}[g \circ u_\beta] = \mathbb{G}_\mu[g \circ \underline{u}_\nu].$$

Letting $\beta \downarrow 0$ in (3.6), we infer that $\underline{u}_\nu$ satisfies (1.14), namely $\underline{u}_\nu$ is a solution of (1.13). If v is a solution of (1.13) then $v \ge \mathbb{K}_\mu[\nu]$ and $g \circ v \in L^1(\Omega; \delta^{\alpha_+})$; consequently $\underline{u}_\nu \le v$ in Ω. □

Proof of Theorem B We first notice that since $g \circ (\mathbb{K}_\mu[\nu]) \in L^1_{\mathrm{loc}}(\Omega)$ and $\mathbb{G}_\mu[g \circ (\mathbb{K}_\mu[\nu])] < \infty$, it follows that $g \circ (\mathbb{K}_\mu[\nu]) \in L^1(\Omega; \delta^{\alpha_+})$ due to a similar argument as in the proof of Theorem A. It is easy to see that $\mathbb{K}_\mu[\nu]$ is a subsolution of (3.1). For $\varrho \in (r_1, r_2)$, we look for a supersolution v of the form

$$v = \varrho\mathbb{K}_\mu[\nu] + c_2\mathbb{G}_\mu[g \circ (\varrho\mathbb{K}_\mu[\nu])] \tag{3.7}$$

where c_2 is the constant in (1.17). By (1.16) and (1.18), we obtain

$$v \le \varrho(1 + c_2c_3\varrho^{-1}\ell(\varrho))\mathbb{K}_\mu[\nu] \quad \text{in } \Omega.$$

The monotonicity property of g implies

$$g \circ v \le g \circ (\varrho(1 + c_2c_3\varrho^{-1}\ell(\varrho))\mathbb{K}_\mu[\nu]) \quad \text{in } \Omega.$$

By (1.16),

$$g \circ v \le \ell(1 + c_2c_3\varrho^{-1}\ell(\varrho))g \circ (\varrho\mathbb{K}_\mu[\nu]) \quad \text{in } \Omega. \tag{3.8}$$

In light of (1.17), we deduce

$$g \circ v \le c_2 g \circ (\varrho\mathbb{K}_\mu[\nu]) = -L_\mu v \tag{3.9}$$

This means v is a supersolution of (3.1).

We apply Theorem 3.2 to derive that problem (1.19) admits a minimal solution $\underline{u}_{\varrho\nu}$ satisfying

$$\varrho\mathbb{K}_\mu[\nu] \le \underline{u}_{\varrho\nu} \le \varrho\mathbb{K}_\mu[\nu] + c_2\mathbb{G}_\mu[g \circ (\mathbb{K}_\mu[\nu])] \quad \text{in } \Omega. \tag{3.10}$$

Estimate (1.20) follows directly from (1.18) and (3.10) with $c_4 = 1 + c_2c_3\varrho^{-1}\ell(\varrho)$.

We next prove (1.21). Due to (1.14), it is sufficient to prove that for ν-a.e. $z \in \partial\Omega$,

$$\lim_{x\to z} \frac{\mathbb{G}_\mu[g \circ \underline{u}_{\varrho\nu}](x)}{\mathbb{K}_\mu[\nu](x)} = 0 \quad \text{non-tangentially.} \tag{3.11}$$

To obtain (3.11), we shall employ Theorem 2.5. Since $\mathbb{K}_\mu[\nu]$ is a positive L_μ-harmonic function satisfying local Harnack inequality, we only need to show that:

(i) $\mathbb{G}_\mu[g \circ \underline{u}_{\varrho\nu}]$ is a positive L_μ-potential;
(ii) $\mathbb{G}_\mu[g \circ \underline{u}_{\varrho\nu}]$ satisfies Harnack inequality.

Since $g \circ \underline{u}_{\varrho\nu} \in L^1(\Omega; \delta^{\alpha_+})$, $\mathrm{tr}^*(\mathbb{G}_\mu[g \circ \underline{u}_{\varrho\nu}]) = 0$ and hence (i) follows from Proposition 2.4 (iv). By (1.20), we infer that $\underline{u}_{\varrho\nu}$ satisfies the local Harnack inequality. Since $\underline{u}_{\varrho\nu}$ can be written under the form (1.14), it follows that $\mathbb{G}_\mu[g \circ \underline{u}_{\varrho\nu}]$ satisfies this inequality too. Hence (ii) is verified. By invoking Theorem 2.5, we get (3.11). □

4 Power source

In this section, we focus on the equation

$$-L_\mu u = u^q \quad \text{in } \Omega. \tag{4.1}$$

4.1 Subcritical case

We start with a lemma the proof of which is an adaptation of an idea in [7].

Lemma 4.1 *Assume* $1 \le q < q^*$ *and* $z \in \partial\Omega$. *Then there exists a constant* $c_{10} = c_{10}(N, \mu, q, \Omega)$ *such that*

$$\mathbb{G}_\mu[K_\mu(\cdot, z)^q](x) \le c_{10}|x - z|^{N+\alpha_+-(N+\alpha_+-2)q} K_\mu(x, z) \quad \forall x \in \Omega. \tag{4.2}$$

Proof The proof is an adaptation of the argument in [7]; for the convenience of the reader it is presented below. By (2.2) and (2.3), there exists a positive constant c_{11} such that for every $x \in \Omega$,

$$\begin{aligned}\mathbb{G}_\mu[K_\mu(\cdot, z)^q](x) \le c_{11}\delta(x)^{\alpha_+} &\int_\Omega |x - y|^{2\alpha_- - N}|y - z|^{(2-\alpha_+-N)q} \\ &\min\left\{|x - y|^{\alpha_+}, |y - z|^{\alpha_+}\right\} dy.\end{aligned} \tag{4.3}$$

Put

$$\begin{aligned}\mathcal{D}_1 &= \Omega \cap B(x, |x - z|/2), \\ \mathcal{D}_2 &= \Omega \cap B(z, |x - z|/2), \\ \mathcal{D}_3 &= \Omega \setminus (\mathcal{D}_1 \cup \mathcal{D}_2),\end{aligned}$$

and

$$\begin{aligned}I_i := \int_{\mathcal{D}_i} &|x - y|^{2\alpha_- - N}|y - z|^{(2-\alpha_+-N)q} \\ &\min\left\{|x - y|^{\alpha_+}, |y - z|^{\alpha_+}\right\} dy, \quad i = 1, 2, 3.\end{aligned}$$

For every $y \in \mathcal{D}_1$, $|x - z| \le 2|y - z|$, therefore

$$\begin{aligned}I_1 &\le c_{12}|x - z|^{(2-\alpha_+-N)q} \int_{\mathcal{D}_1} |x - y|^{1+\alpha_- - N} dy \\ &\le c'_{12}|x - z|^{1+\alpha_- -(N+\alpha_+-2)q}.\end{aligned} \tag{4.4}$$

For every $y \in \mathcal{D}_2$, $|x - z| \le 2|x - y|$, hence

$$\begin{aligned}I_2 &\le c_{13}|x - z|^{1+\alpha_- - N} \int_{\mathcal{D}_2} |y - z|^{(2-\alpha_+-N)q} dy \\ &\le c'_{13}|x - z|^{1+\alpha_- -(N+\alpha_+-2)q}.\end{aligned} \tag{4.5}$$

For every $y \in \mathcal{D}_3$, $|y-z| \leq 3|x-y|$, therefore

$$I_3 \leq c_{14} \int_{\mathcal{D}_3} |y-z|^{1+\alpha_- - N - (N+\alpha_+-2)q} dy \leq c'_{14} |x-z|^{1+\alpha_- - (N+\alpha_+-2)q}. \tag{4.6}$$

Combining (4.3)–(4.6), we obtain

$$\mathbb{G}_\mu[K_\mu(\cdot, z)^q](x) \leq c_{11}(c'_{12} + c'_{13} + c'_{14})\delta(x)^{\alpha_+} |x-z|^{1+\alpha_- - (N+\alpha_+-2)q}. \tag{4.7}$$

Estimate (4.2) follows straightforward from (2.3) and (4.7). □

Proposition 4.2 *Assume $0 < q < q^*$ and ν is a positive finite measure on $\partial\Omega$. Then $\mathbb{K}_\mu[\nu] \in L^q(\Omega; \delta^{\alpha_+})$ and there exists a constant $c_{15} = c_{15}(N, \mu, q, \Omega)$ such that*

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q] \leq c_{15} \, \|\nu\|^{q-1}_{\mathfrak{M}(\partial\Omega)} \, \mathbb{K}_\mu[\nu] \quad \textit{in } \Omega. \tag{4.8}$$

Proof We may assume that $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$ (if it is not the case, one can replace ν by $\nu / \|\nu\|_{\mathfrak{M}(\partial\Omega)}$). We first consider the case $q \geq 1$. From (2.5) and the fact that $L^{q^*}_w(\Omega; \delta^{\alpha_+}) \subset L^q(\Omega; \delta^{\alpha_+})$, we deduce that $\mathbb{K}_\mu[\nu] \in L^q(\Omega; \delta^{\alpha_+})$. It follows from (1.6) and Jensen's inequality that

$$\mathbb{K}_\mu[\nu](x)^q \leq \int_{\partial\Omega} K_\mu(x,z)^q d\nu(z) \quad \text{for a.e. } x \in \Omega.$$

Consequently,

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q](x) \leq \int_{\partial\Omega} \int_\Omega G_\mu(x,y) K_\mu(y,z)^q d\nu(z) dy.$$

By Lemma 4.1, since $N + \alpha_+ - (N + \alpha_+ - 2)q > 0$,

$$\begin{aligned}\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q](x) &\leq c_{10} \int_{\partial\Omega} |x-z|^{N+\alpha_+-(N+\alpha_+-2)q} K_\mu(x,z) d\nu(z) \\ &\leq c_{10} (\mathrm{diam}(\Omega))^{N+\alpha_+-(N+\alpha_+-2)q} \mathbb{K}_\mu[\nu](x).\end{aligned}$$

Thus we obtain (4.8).

If $0 < q < 1$ then

$$\mathbb{G}_\mu\left[\mathbb{K}^q_\mu[\nu]\right] \leq \mathbb{G}_\mu\left[1 + \mathbb{K}_\mu[\nu]\right] = \mathbb{G}_\mu[1] + \mathbb{G}_\mu\left[\mathbb{K}_\mu[\nu]\right] \quad \text{in } \Omega.$$

From the case $q = 1$, we deduce that

$$\mathbb{G}_\mu[\mathbb{K}^q_\mu[\nu]] \leq \mathbb{G}_\mu[1] + c_{15}\mathbb{K}_\mu[\nu] \quad \text{in } \Omega.$$

By the estimate $\mathbb{G}_\mu[1] \leq c_{16}\mathbb{K}_\mu[\nu]$, where $c_{16} = c_{16}(N, \mu, \Omega)$, we conclude (4.8). □

Lemma 4.3 *Let $f \in L^1(\Omega; \delta^{\alpha_+})$, $f \geq 0$, $\nu \in \mathfrak{M}^+(\partial\Omega)$, $\nu \not\equiv 0$ and $\phi \in C^1([0,\infty))$ be a concave, nondecreasing function such that $\phi(1) \geq 0$ and ϕ' is bounded. Let φ be a positive function in $L^1_{loc}(\Omega)$ such that $-L_\mu\varphi \geq f$. Then*

$$\phi'\left(\frac{\varphi}{\mathbb{K}_\mu[\nu]}\right) f \in L^1(\Omega; \delta^{\alpha_+}), \tag{4.9}$$

$$-L_\mu\left[\mathbb{K}_\mu[\nu]\phi\left(\frac{\varphi}{\mathbb{K}_\mu[\nu]}\right)\right] \geq \phi'\left(\frac{\varphi}{\mathbb{K}_\mu[\nu]}\right) f \quad \textit{in the weak sense.} \tag{4.10}$$

Proof Since $-L_\mu \varphi \geq f \geq 0$, by Proposition 2.4, there exist $\tau \in \mathfrak{M}^+(\Omega; \delta^{\alpha_+})$ and $\lambda \in \mathfrak{M}^+(\partial\Omega)$ such that

$$\varphi = \mathbb{G}_\mu[f] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\lambda].$$

Put $\psi := \mathbb{K}_\mu[\nu]$. Let $\{f_n\}$ and $\{\tau_n\}$ be two sequences in $C_c^\infty(\Omega)$ such that $\{f_n\}$ converges to f in $L^1(\Omega; \delta^{\alpha_+})$ and $\{\tau_n\}$ converges to τ in the weak sense of $\mathfrak{M}^+(\Omega; \delta^{\alpha_+})$. Let $\{\nu_n\}$ and $\{\lambda_n\}$ be two sequences in $C^1(\partial\Omega)$ converging to ν and λ respectively in the weak sense of $\mathfrak{M}^+(\partial\Omega)$. Put $\varphi_n := \mathbb{G}_\mu[f_n] + \mathbb{G}_\mu[\tau_n] + \mathbb{K}_\mu[\lambda_n]$ and $\psi_n := \mathbb{K}_\mu[\nu_n]$. By the bootstrap argument, one can prove that $\varphi_n, \psi_n \in C^2(\Omega)$ for every $n \in \mathbb{N}$. By [20] $\{\mathbb{G}_\mu[f_n]\}$, $\{\mathbb{G}_\mu[\tau_n]\}$, $\{\mathbb{K}_\mu[\lambda_n]\}$ and $\{\mathbb{K}_\mu[\nu_n]\}$ converge to $\mathbb{G}_\mu[f]$, $\mathbb{G}_\mu[\tau]$, $\mathbb{K}_\mu[\lambda]$ and $\mathbb{K}_\mu[\nu]$ respectively in $L^1(\Omega, \delta^{\alpha_+})$. As a consequence, up to subsequences, $\{\varphi_n\}$ and $\{\psi_n\}$ converge to φ and ψ respectively a.e. in Ω. Therefore, for n large enough, $\psi_n > 0$.

Due to [10, Lemma 5.3],

$$-\Delta\left[\psi_n \phi\left(\frac{\varphi_n}{\psi_n}\right)\right] \geq \phi'\left(\frac{\varphi_n}{\psi_n}\right)(-\Delta\varphi_n) + \left[\phi\left(\frac{\varphi_n}{\psi_n}\right) - \frac{\varphi_n}{\psi_n}\phi'\left(\frac{\varphi_n}{\psi_n}\right)\right](-\Delta\psi_n).$$

It follows that

$$-L_\mu\left[\psi_n \phi\left(\frac{\varphi_n}{\psi_n}\right)\right] \geq \phi'\left(\frac{\varphi_n}{\psi_n}\right)(-L_\mu\varphi_n) + \left[\phi\left(\frac{\varphi_n}{\psi_n}\right) - \frac{\varphi_n}{\psi_n}\phi'\left(\frac{\varphi_n}{\psi_n}\right)\right](-L_\mu\psi_n).$$

Consequently,

$$-L_\mu\left[\psi_n \phi\left(\frac{\varphi_n}{\psi_n}\right)\right] \geq \phi'\left(\frac{\varphi_n}{\psi_n}\right) f_n. \tag{4.11}$$

Then for every nonnegative function $\zeta \in X(\Omega)$, there holds

$$-\int_\Omega \psi_n \phi\left(\frac{\varphi_n}{\psi_n}\right) L_\mu \zeta \, dx \geq \int_\Omega \phi'\left(\frac{\varphi_n}{\psi_n}\right) f_n \zeta \, dx. \tag{4.12}$$

We see that

$$0 \leq \psi_n \phi\left(\frac{\varphi_n}{\psi_n}\right) \leq \psi_n \left(\phi(0) + \phi'(0)\frac{\varphi_n}{\psi_n}\right) = c_{17}(\psi_n + \varphi_n). \tag{4.13}$$

By (2.4) and (2.5), $\{\varphi_n\}$ and $\{\psi_n\}$ are uniformly bounded in $L^p(\Omega; \delta^{-\alpha_-})$ for $p \in (1, \frac{N-\alpha_-}{N-1-\alpha_-})$. Due to Hölder inequality, $\{\varphi_n\}$ and $\{\psi_n\}$ are uniformly integrable with respect to $\delta^{\alpha_-}dx$. In view of Vitali theorem $\{\varphi_n\}$ and $\{\psi_n\}$ converge to φ and ψ in $L^1(\Omega; \delta^{-\alpha_-})$ respectively. By (4.13) and dominated convergence theorem we deduce that

$$\psi_n \phi\left(\frac{\varphi_n}{\psi_n}\right) \to \psi\phi\left(\frac{\varphi}{\psi}\right) \quad \text{in } L^1(\Omega; \delta^{-\alpha_-}).$$

Due to Fatou lemma, by sending $n \to \infty$ in (4.12), we obtain (4.9) and (4.10). □

Theorem 4.4 *Let $q > 1$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$, $\nu \not\equiv 0$. If problem (D_ν) admits a positive weak solution then*

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q] \leq \frac{1}{q-1}\mathbb{K}_\mu[\nu] \quad \textit{in } \Omega. \tag{4.14}$$

Proof Let u a positive weak solution of (D_ν); then by Theorem A, $u^q \in L^1(\Omega;\delta^{\alpha_+})$ and (1.14) holds. Consequently $\mathbb{K}_\mu[\nu]^q \in L^1(\Omega;\delta^{\alpha_+})$. Now applying Lemma 4.3 with $f = u^q, \varphi = u$ and

$$\phi(s) = \begin{cases} \dfrac{1-s^{1-q}}{q-1}, & \text{if } s \geq 1, \\ s-1 & \text{if } 0 \leq s < 1, \end{cases}$$

we obtain the following estimate in the weak sense

$$-L_\mu\left[\mathbb{K}_\mu[\nu]\phi\left(\frac{u}{\mathbb{K}_\mu[\nu]}\right)\right] \geq \left(\frac{u}{\mathbb{K}_\mu[\nu]}\right)^{-q} u^q = \mathbb{K}_\mu[\nu]^q. \tag{4.15}$$

Put

$$\Psi := \mathbb{K}_\mu[\nu]\phi\left(\frac{u}{\mathbb{K}_\mu[\nu]}\right) \quad \text{and} \quad \tilde{\Psi} := \mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q].$$

Then Ψ is an L_μ-superharmonic function and by Proposition 2.4, Ψ admits a nonnegative normalized boundary trace. By Kato lemma (see [23]), $(\tilde{\Psi}-\Psi)_+$ is an L_μ-subharmonic function and $\operatorname{tr}^*((\tilde{\Psi}-\Psi)_+) = 0$. It follows that $(\tilde{\Psi}-\Psi)_+ = 0$ and hence $\tilde{\Psi} \leq \Psi$ in Ω. This means

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^q] \leq \mathbb{K}_\mu[\nu]\phi\left(\frac{u}{\mathbb{K}_\mu[\nu]}\right) \leq \frac{1}{q-1}\mathbb{K}_\mu[\nu].$$

□

Proof of Theorem C The theorem follows from Lemma 4.1 and Theorem 4.4. □

Proposition 4.5 *Assume $0 < q < q^*$, $q \neq 1$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$.*

(i) *If $q > 1$ then there exists a positive number $\varrho_0 > 0$ depending on N, μ, q, Ω such that for every $\varrho \in (0, \varrho_0)$ problem $(D_{\varrho\nu})$ admits a minimal weak solution $\underline{u}_{\varrho\nu}$.*
(ii) *If $q \in (0,1)$ then for every $\varrho > 0$ problem $(D_{\varrho\nu})$ admits a minimal weak solution $\underline{u}_{\varrho\nu}$.*

For any $1 \neq q \in (0, q^*)$, $\underline{u}_{\varrho\nu}$ satisfies (1.20) and (1.21).

Proof We shall apply Theorem B to deduce the existence of a solution of $(D_{\varrho\nu})$. One can verify that the functions $g(x,s) = s^q$ and $\ell(s) = s^q$ with $q > 0$ satisfy (1.16). From Proposition 4.2 we deduce that condition (1.18) is fulfilled with the constant c_{15}. For such g and ℓ, condition (1.17) is valid if one can find a positive constant c_{18} such that

$$1 + c_{18}c_{15}\varrho^{q-1} \leq c_{18}^{\frac{1}{q}}. \tag{4.16}$$

If $q > 1$ then there exist $\varrho_0 = \varrho_0(q, c_{15})$ and $c_{18} = c_{18}(q)$ such that (4.16) holds true for every $\varrho \in (0, \varrho_0)$. If $q < 1$ then for every $\varrho \in [1, \infty)$ one can choose $c_{18} = c_{18}(c_{15})$ large enough such that (4.16) holds. If $q < 1$ then for every $\varrho \in (0,1)$ one can choose $c_{18} = c_{18}(q, \varrho, c_{15})$ large enough such that (4.16) holds. Hence, by Theorem B, there exists a minimal solution $\underline{u}_{\varrho\nu}$ of $(D_{\varrho\nu})$ which satisfies (1.20) and (1.21). □

Lemma 4.6 *Let $0 < q \neq 1$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Then there is a constant $c_{19} = c_{19}(N, \mu, q, \Omega)$ such that if u is a solution of (D_ν)*

$$\|u\|_{L^1(\Omega;\delta^{-\alpha_-})} + \left\|u^q\right\|_{L^1(\Omega;\delta^{\alpha_+})} \leq c_{19}(1 + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{4.17}$$

Proof Indeed, by taking $\zeta = \varphi_{\mu,1}$ (the first eigenfunction of $-L_\mu$) in the formulation satisfied by u, we obtain

$$\lambda_{\mu,1}\int_\Omega u\varphi_{\mu,1}dx = \int_\Omega u^q\varphi_{\mu,1}dx + \lambda_{\mu,1}\int_\Omega \mathbb{K}_\mu[\nu]\varphi_{\mu,1}dx. \tag{4.18}$$

Case 1: $q > 1$. By Young inequality, we get

$$\int_\Omega u\varphi_{\mu,1}dx \le (2\lambda_{\mu,1})^{-1}\int_\Omega u^q\varphi_{\mu,1}dx + (2\lambda_{\mu,1})^{\frac{1}{q-1}}\int_\Omega \varphi_{\mu,1}dx. \tag{4.19}$$

By (4.18) and (4.19), we obtain

$$\int_\Omega u^q\varphi_{\mu,1}dx + 2\lambda_{\mu,1}\int_\Omega \mathbb{K}_\mu[\nu]\varphi_{\mu,1}dx \le (2\lambda_{\mu,1})^{\frac{q}{q-1}}\int_\Omega \varphi_{\mu,1}dx. \tag{4.20}$$

Since the second term on the left hand-side of (4.20) is nonnegative, we deduce by (1.8) that

$$\|u\|^q_{L^q(\Omega;\delta^{\alpha_+})} \le c_1^2(2\lambda_{\mu,1})^{\frac{q}{q-1}}\int_\Omega \delta^{\alpha_+}dx \le c_{20}. \tag{4.21}$$

On the other hand, we derive from (1.14), (2.4) and (2.5) that

$$\|u\|_{L^1(\Omega;\delta^{-\alpha_-})} \le c_{21}(\left\|u^q\right\|_{L^1(\Omega;\delta^{\alpha_+})} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{4.22}$$

Combining (4.21) and (4.22), we obtain (4.17).
Case 2: $q \in (0,1)$. By Young inequality, we have

$$\int_\Omega u^q\varphi_{\mu,1}dx \le \frac{\lambda_{\mu,1}}{2}\int_\Omega u\varphi_{\mu,1}dx + (2\lambda_{\mu,1}^{-1})^{\frac{q}{1-q}}\int_\Omega \varphi_{\mu,1}dx.$$

Consequently,

$$\int_\Omega u\varphi_{\mu,1}dx \le (2\lambda_{\mu,1}^{-1})^{\frac{1}{1-q}}\int_\Omega \varphi_{\mu,1}dx + 2\int_\Omega \mathbb{K}_\mu[\nu]\varphi_{\mu,1}dx.$$

Therefore

$$\|u\|_{L^1(\Omega;\delta^{\alpha_+})} \le c_{22}(1 + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{4.23}$$

Combining (4.22) and (4.23) leads to (4.17). □

Theorem 4.7 *Assume $q \in (1, q^*)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. Then there exists a threshold value $\varrho^* \in \mathbb{R}_+$ for $(D_{\varrho\nu})$ such that the following holds.*

- (i) *If $\varrho \in (0, \varrho^*]$ then $(D_{\varrho\nu})$ admits a minimal weak solution $\underline{u}_{\varrho\nu}$. If $\varrho \in (0,\varrho^*)$ then $\underline{u}_{\varrho\nu}$ satisfies (1.20) and (1.21). Moreover $\{\underline{u}_{\varrho\nu}\}$ is an increasing sequence which converges, as $\varrho \to \varrho^*$, to the minimal positive weak solution $\underline{u}_{\varrho^*\nu}$ of $(D_{\varrho^*\nu})$ in $L^1(\Omega;\delta^{-\alpha_-})$ and in $L^1(\Omega;\delta^{\alpha_+})$.*
- (ii) *If $\varrho > \varrho^*$ then there exists no positive weak solution of $(D_{\varrho\nu})$.*

Proof Put

$$\mathcal{A} := \{\varrho > 0 : (D_{\varrho\nu}) \text{ admits a weak solution}\} \quad\text{and}\quad \varrho^* := \sup\mathcal{A}.$$

By Proposition 4.5, $(D_{\varrho\nu})$ admits a solution for $\varrho > 0$ small, therefore $\mathcal{A} \neq \emptyset$. Moreover, from Theorem 4.4, we deduce that ϱ^* is finite.

We shall show that $(0,\varrho^*) \subset \mathcal{A}$. To this purpose, we have to show that if $0 < \varrho < \varrho'$ and $\mathcal{A} \ni \varrho' < \varrho^*$ then $\varrho \in \mathcal{A}$. Since $\varrho' \in \mathcal{A}$, due to Theorem 4.4, there exists a minimal weak solution $\underline{u}_{\varrho'\nu}$ of $(D_{\varrho'\nu})$ which is greater than $\varrho\mathbb{K}_\mu[\nu]$. By Theorem 3.2, $(D_{\varrho\nu})$ admits a minimal weak solution $\underline{u}_{\varrho\nu}$, i.e. $\varrho \in \mathcal{A}$.

Next we prove that $\varrho^* \in \mathcal{A}$, namely problem $(D_{\varrho^*\nu})$ admits a weak solution. Let $\{\varrho_n\}$ be an increasing sequence converging to ϱ^*. For each n, let $u_{\varrho_n\nu}$ be a weak solution of $(D_{\varrho_n\nu})$. Then $u_{\varrho_n\nu} \in L^1(\Omega;\delta^{-\alpha_-}) \cap L^q(\Omega;\delta^{\alpha_+})$ and it satisfies

$$-\int_\Omega u_{\varrho_n\nu} L_\mu\zeta\,dx = \int_\Omega u_{\varrho_n\nu}^q \zeta\,dx - \varrho_n \int_\Omega \mathbb{K}_\mu[\nu] L_\mu\zeta\,dx \quad \forall \zeta \in X(\Omega). \tag{4.24}$$

It follows from Lemma 4.6 that the sequence $\{u_{\varrho_n\nu}^q\}$ is uniformly bounded in $L^1(\Omega;\delta^{\alpha_+})$ and hence by local regularity for elliptic equations there exists a subsequence, still denoted by the same notation, such that $\{u_{\varrho_n\nu}\}$ converges a.e. to a function $u_{\varrho^*\nu}$. From Theorem A, there holds

$$u_{\varrho_n\nu} = \mathbb{G}_\mu[u_{\varrho_n\nu}^q] + \varrho_n\mathbb{K}_\mu[\nu]. \tag{4.25}$$

Thanks to Proposition 2.1, $\{u_{\varrho_n\nu}\}$ is uniformly bounded in $L^{q_1}(\Omega;\delta^{-\alpha_-})$ and in $L^{q_2}(\Omega;\delta^{\alpha_+})$ where $1 < q_1 < \frac{N-\alpha_-}{N-1-\alpha_-}$ and $q < q_2 < q^*$. We invoke Holder inequality to infer that $\{u_{\varrho_n\nu}\}$ and $\{u_{\varrho_n\nu}^q\}$ are uniformly integrable with respect to $\delta^{-\alpha_-}dx$ and $\delta^{\alpha_+}dx$ respectively. As a consequence, $\{u_{\varrho_n\nu}\}$ converges to $u_{\varrho^*\nu}$ in $L^1(\Omega;\delta^{-\alpha_-})$ and $\{u_{\varrho_n\nu}^q\}$ converges to $u_{\varrho^*\nu}^q$ in $L^1(\Omega;\delta^{\alpha_+})$. Letting $n\to\infty$ in (4.24) implies

$$-\int_\Omega u_{\varrho^*\nu} L_\mu\zeta\,dx = \int_\Omega u_{\varrho^*\nu}^q \zeta\,dx - \varrho^* \int_\Omega \mathbb{K}_\mu[\nu] L_\mu\zeta\,dx \quad \forall \zeta \in X(\Omega). \tag{4.26}$$

We infer from Theorem A that $u_{\varrho^*\nu}$ is a solution of $(D_{\varrho^*\nu})$.

Notice that, in light of Theorem 3.2 and the above argument, one can prove that $\{\underline{u}_{\varrho\nu}\}$ is an increasing sequence converging to the minimal solution $\underline{u}_{\varrho^*\nu}$ of $(D_{\varrho^*\nu})$ in $L^1(\Omega;\delta^{-\alpha_-})$ and in $L^q(\Omega;\delta^{\alpha_+})$.

We next show that for each $\varrho \in (0,\varrho^*)$, there exists a minimal weak solution $\underline{u}_{\varrho\nu}$ of $(D_{\varrho\nu})$ which satisfies (1.20). Take $\varrho' = \frac{\varrho+\varrho^*}{2}$ and let $u_{\varrho'\nu}$ be a solution of $(D_{\varrho'\nu})$. We apply (4.10) with ν replaced by $\varrho'\nu$, $\varphi = u_{\varrho'\nu}$, $f = u_{\varrho'\nu}^q$ and

$$\phi(s) = \begin{cases} s(1+\varepsilon s^{q-1})^{-\frac{1}{q-1}}, & \text{if } s \ge 1,\\ \left(\dfrac{\varrho}{\varrho'}\right)^q s + \left(\dfrac{\varrho}{\varrho'}\right) - \left(\dfrac{\varrho}{\varrho'}\right)^q & \text{if } 0 \le s \le 1 \end{cases} \quad \text{with} \quad \varepsilon = \left(\frac{\varrho'}{\varrho}\right)^{q-1} - 1.$$

We get

$$\begin{aligned} -L_\mu\left(\mathbb{K}_\mu[\varrho'\nu]\phi\left(\frac{u_{\varrho'\nu}}{\mathbb{K}_\mu[\varrho'\nu]}\right)\right) &\ge \phi'\left(\frac{u_{\varrho'\nu}}{\mathbb{K}_\mu[\varrho'\nu]}\right) u_{\varrho'\nu}^q \\ &= \left(\mathbb{K}_\mu[\varrho'\nu]\phi\left(\frac{u_{\varrho'\nu}}{\mathbb{K}_\mu[\varrho'\nu]}\right)\right)^q. \end{aligned}$$

Therefore

$$\Psi = \mathbb{K}_\mu[\varrho'\nu]\phi\left(\frac{u_{\varrho'\nu}}{\mathbb{K}_\mu[\varrho'\nu]}\right)$$

is a supersolution of (4.1). Moreover $\Psi \ge \varrho\mathbb{K}_\mu[\nu]$. By Theorem 3.2 there exists a minimal weak solution $\underline{u}_{\varrho\nu}$ of $(D_{\varrho\nu})$ such that

$$\varrho\mathbb{K}_\mu[\nu] \le \underline{u}_{\varrho\nu} \le \Psi \quad \text{in } \Omega.$$

This implies

$$\varrho \mathbb{K}_\mu[\nu] \leq \underline{u}_{\varrho\nu} \leq \varepsilon^{-\frac{1}{q-1}} \varrho' \mathbb{K}_\mu[\nu] \quad \text{in } \Omega.$$

Therefore we get (1.20) with $c_4 = \varrho^{-1}\varrho'\varepsilon^{-\frac{1}{q-1}}$. Finally, (1.21) can be obtained by a similar argument as in the proof of Theorem B. □

Proof of Theorem D Part (1) follows from Theorem 4.7. Part (2) follows from Proposition 4.5 (ii).

If $\nu = \varrho\delta_z$, by (1.14) and (1.20), we obtain

$$\varrho \leq \frac{u_{\varrho\delta_z}(x)}{K_\mu(x,z)} \leq \varrho + c_{23}\frac{\mathbb{G}_\mu[K_\mu(\cdot,z)^q](x)}{K_\mu(x,z)} \tag{4.27}$$

Since $q < q^*$, it follows from Lemma 4.1 that

$$\lim_{x\to z} \frac{\mathbb{G}_\mu[K_\mu(\cdot,z)^q](x)}{K_\mu(x,z)} = 0.$$

Thus, by (4.27), we conclude (1.24). □

Proof of Proposition E If $q > 1$, assumption (1.25) guarantees the existence of a solution u_{ν_n} of (D_{ν_n}). Moreover, since $\{\nu_n\}$ converges weakly to ν, it follows that $\|\nu\|_{\mathfrak{M}(\partial\Omega)} \leq \varrho^*$. Due to Lemma 4.6, the sequence $\{u_{\nu_n}\}$ is uniformly bounded in $L^q(\Omega;\delta^{\alpha_+})$. Employing a similar argument as in the proof of Theorem 4.7, we obtain the convergence in $L^1(\Omega;\delta^{-\alpha_-})$ and in $L^q(\Omega;\delta^{\alpha_+})$.

If $q \in (0,1)$, due to Lemma 4.6, we obtain the convergence in $L^1(\Omega;\delta^{-\alpha_-})$. □

We next consider the case $q = 1$.

Lemma 4.8 *Let $\kappa > 0$ and u be a positive solution of*

$$-L_\mu u = \kappa u \quad \text{in } \Omega. \tag{4.28}$$

Then u satisfies the Harnack inequality; i.e. for every $a \in (0,1)$ and $x \in \Omega$,

$$\sup_{B(x,a\delta(x))} u \leq c_{24} \inf_{B(x,a\delta(x))} u \tag{4.29}$$

where $c_{24} = c_{24}(N,\mu,q,\Omega)$.

Proof Equation (4.28) can be written as follows

$$-\Delta u = \left(\frac{\mu}{\delta^2} + \kappa\right) u \qquad \text{in } \Omega. \tag{4.30}$$

Take arbitrarily $a \in (0,1)$ and $x_0 \in \Omega$. Put $d := \frac{a+1}{2}\delta(x_0)$ and $M := \max_{B(x_0,d)} u$. Put

$$y_0 := d^{-1}x_0, \quad \Omega^d := d^{-1}\Omega, \quad \delta_d(y) := \mathrm{dist}\,(y, \partial\Omega^d) \text{ with } y \in \Omega^d.$$

We define

$$v_d(y) := M^{-1}u(dy), \quad \forall y \in \Omega^d.$$

Clearly, $\max_{B(y_0,1)} v_d = 1$ and due to (4.30) we deduce that v_d is a solution of

$$-\Delta v_d = V v_d \qquad \text{in } \Omega^d. \tag{4.31}$$

where

$$V(y) := \frac{\mu}{\delta_d(y)^2} + d^2\kappa.$$

One can find a positive constant c_{25} such that $V(y) \le c_{25}\delta_d(y)^{-2}$ for every $y \in \Omega^d$. Notice that $B(y_0, 1) \subset \Omega^d$ and for every $y \in B(y_0, 1)$, there holds

$$\delta_d(y) \ge \frac{1-a}{1+a}.$$

Hence $0 \le V \le c_{26}$ in $B(y_0, 1)$ where $c_{26} = c_{26}(a, \mu)$. By applying Harnack inequality, we deduce that there is a constant $c_{27} = c_{27}(a, \mu, N, \Omega)$ such that

$$\sup_{B(y_0, \frac{2a}{a+1})} v_d \le c_{27} \inf_{B(y_0, \frac{2a}{a+1})} v_d.$$

Thus we obtain (4.29). □

Proof of Theorem F Put

$$\kappa_0 = \min\left\{1, \left\|\mathbb{G}_\mu[1]\right\|^{-1}_{L^\infty(\Omega)}\right\}.$$

□

Claim 1 *For any $\kappa \in (0, \kappa_0)$ there exists a minimal solution $\underline{u}_{\kappa,\nu}$ of (E^κ_ν).*

Fix $q \in (1, q^*)$ such that $\kappa < (q\left\|\mathbb{G}_\mu[1]\right\|_{L^\infty(\Omega)})^{-1}$ and let ϱ^* be the threshold value for $(D_{\varrho\nu})$. Put $\varrho = \|\nu\|_{\mathfrak{M}(\partial\Omega)} > 0$.

We first assume that $\varrho \in (0, \varrho^*)$ and let $\underline{u}_\nu$ be the minimal weak solution of (D_ν). By Young inequality, we get

$$\underline{u}_\nu^q + 1 \ge \underline{u}_\nu + \frac{1}{q} \ge \kappa(\underline{u}_\nu + \mathbb{G}_\mu[1]) \quad \text{in } \Omega.$$

It follows that

$$-L_\mu(\underline{u}_\nu + \mathbb{G}_\mu[1]) \ge \kappa(\underline{u}_\nu + \mathbb{G}_\mu[1]) \quad \text{in } \Omega.$$

Therefore $\underline{u}_\nu + \mathbb{G}_\mu[1]$ is a super solution of the equation

$$-L_\mu u = \kappa u \quad \text{in } \Omega. \tag{4.32}$$

Clearly $\mathbb{K}_\mu[\nu]$ is a subsolution of (4.32). By Theorem 3.2 there is a minimal weak solution $\underline{u}_{\kappa,\nu}$ of (E^κ_ν) which satisfies $\mathbb{K}_\mu[\nu] \le \underline{u}_{\kappa,\nu} \le \underline{u}_\nu + \mathbb{G}_\mu[1]$ in Ω. Since $\mathbb{G}_\mu[1] \le c_{16}\mathbb{K}_\mu[\nu]$, we infer that $\underline{u}_{\kappa,\nu}$ satisfies (1.20) and (1.21).

If $\varrho \ge \varrho^*$ then there exists $m > 0$ such that $\varrho/m \in (0, \varrho^*)$. Let $\underline{u}_{\kappa,\frac{\nu}{m}}$ be the minimal weak solution of $(E^\kappa_{\frac{\nu}{m}})$. Put $\underline{u}_{\kappa,\nu} = m\underline{u}_{\kappa,\frac{\nu}{m}}$ then by the linearity, we deduce that $\underline{u}_{\kappa,\nu}$ is the minimal weak solution of (E^κ_ν) with satisfies (1.20) and (1.21).

Claim 2 *There exists a number $\kappa^* \in (0, \lambda_{\mu,1}]$ such that the following holds.*

(i) *If $\kappa \in (0, \kappa^*)$ then (E^κ_ν) admits a solution;*
(ii) *If $\kappa > \kappa^*$ then (E^κ_ν) admits no solution.*

Put $\mathcal{B} := \{\kappa > 0 : (E_\nu^\kappa)$ admits a weak solution$\}$ and denote $\kappa^* := \sup \mathcal{B}$. We shall show that $(0, \kappa^*) \subset \mathcal{B}$. Take $\kappa' \in \mathcal{B}$ and let $u_{\kappa',\nu}$ be the minimal solution of $(E_\nu^{\kappa'})$. For any $\kappa \in (0, \kappa')$, $u_{\kappa',\nu}$ and $\mathbb{K}_\mu[\nu]$ are respectively super and sub solutions of (E_ν^κ) such that $\mathbb{K}_\mu[\nu] \le u_{\kappa',\nu}$. Then by Theorem 3.2 there exists a minimal solution $\underline{u}_{\kappa,\nu}$ of (E_ν^κ) satisfying $\mathbb{K}_\mu[\nu] \le \underline{u}_{\kappa,\nu} \le u_{\kappa',\nu}$ in Ω. Hence $\kappa \in \mathcal{B}$.

By Lemma 4.8, $\mathbb{G}_\mu[\underline{u}_{\kappa,\nu}]$ satisfies local Harnack inequality. Hence, we deduce from Theorem 2.5 that, for ν-a.e. $z \in \partial\omega$, there holds

$$\lim_{x \to z} \frac{\mathbb{G}_\mu[\underline{u}_{\kappa,\nu}](x)}{\mathbb{K}_\mu[\nu](x)} = 0.$$

Consequently, (1.21) remains valid with $\underline{u}_{\varrho\nu}$ replaced by $\underline{u}_{\kappa,\nu}$.

Now let $\nu \in \mathfrak{M}^+(\partial\Omega)$, $\kappa \in \mathcal{B}$ and denote by $u_{\kappa,\nu}$ a solution of (E_ν^κ). Then by Theorem A,

$$-\int_\Omega u_{\kappa,\nu} L_\mu \zeta dx = \kappa \int_\Omega u_{\kappa,\nu} \zeta dx - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta dx \quad \forall \zeta \in X(\Omega).$$

Taking $\zeta = \varphi_{\mu,1}$, we obtain

$$\lambda_{\mu,1} \int_\Omega u_{\kappa,\nu} \varphi_{\mu,1} dx = \kappa \int_\Omega u_{\kappa,\nu} \varphi_{\mu,1} dx + \lambda_{\mu,1} \int_\Omega \mathbb{K}_\mu[\nu] \varphi_{\mu,1} dx, \tag{4.33}$$

which implies that $\kappa < \lambda_{\mu,1}$. Consequently, $\kappa^* \le \lambda_{\mu,1}$.

We show that $\lambda_{\mu,1} \notin \mathcal{B}$ by contradiction. Indeed, suppose that there exists $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that the problem

$$-L_\mu u = \lambda_{\mu,1} u \quad \text{in } \Omega, \qquad \operatorname{tr}^*(u) = \nu \tag{4.34}$$

admits a weak solution $\hat{u}$. Take $\varphi_{\mu,1}$ as a test function in the weak formulation satisfied by $\hat{u}$, we deduce $\nu \equiv 0$, which is a contradiction.

Now let $\kappa \in (0, \kappa^*)$ and assume $\{\nu_n\}$ is a sequence of measures in $\mathfrak{M}^+(\partial\Omega)$ which converges weakly to $\nu \in \mathfrak{M}^+(\partial\Omega)$. Let u_{κ,ν_n} be a solution of $(E_{\nu_n}^\kappa)$. By (4.33), we deduce

$$\left\| u_{\kappa,\nu_n} \right\|_{L^1(\Omega;\delta^{\alpha_+})} \le c_{28} (\lambda_{\mu,1} - \kappa)^{-1} \|\nu_n\|_{\mathfrak{M}(\partial\Omega)} \le c_{29} (\lambda_{\mu,1} - \kappa)^{-1} \|\nu\|_{\mathfrak{M}(\partial\Omega)} .$$

By a similar argument as in the proof of Theorem 4.7, we deduce that, up to a subsequence, $\{u_{\kappa,\nu_n}\}$ converges to a solution $u_{\kappa,\nu}$ of (E_ν^κ) in $L^1(\Omega, \delta^{-\alpha_-})$.

Remark (i) If $\kappa > 0$ small then $\underline{u}_{\kappa,\nu}$ satisfies (1.20). Moreover, if $\nu = \varrho\delta_z$ with $\varrho > 0$, $z \in \partial\Omega$ then $\underline{u}_{\kappa,\varrho\delta_z}$ satisfies (1.24).

(ii) It is notified by the referee that $\kappa^* = \lambda_{\mu,1}$. This can be obtained by noticing that $-L_\mu - \kappa$ admits the Green function $G_{\mu,\kappa}$ for any $\kappa < \mu$ and then by proving a modification of Proposition 2.4 for $G_{\mu,\kappa}$ (see [24] for the existence of the Green function $G_{\mu,\kappa}$). The weaker statement $\kappa^* \le \lambda_{\mu,1}$ in the present paper is essentially in order to simplify the proofs and to streamline the exposition.

4.2 Supercritical case

Proof of Theorem G This theorem is a consequence of a more general result established in [8]. We present below a simple proof for the special case treated here.

Suppose by contradiction that for some $\varrho > 0$ and $z \in \partial\Omega$ there exists a positive weak solution u of $(D_{\varrho\delta_z})$. Then by Theorem A, $u \in L^q(\Omega; \delta^{\alpha_+})$ and $u \ge \varrho K_\mu(\cdot, z)$. This, along with (2.3), implies

$$\begin{aligned}\int_\Omega \delta(x)^{\alpha_+} u(x)^q dx &\geq \int_\Omega \delta(x)^{\alpha_+} K_\mu(x,z)^q dx \\ &\geq \int_\Omega \delta(x)^{\alpha_+(q+1)} |x-y|^{(2\alpha_- - N)q} dx \\ &\geq c_{30} \int_{\left\{x\in\Omega:\delta(x)\geq\frac{1}{2}|x-y|\right\}} \delta(x)^{\alpha_+(q+1)} |x-y|^{(2\alpha_- - N)q} dx.\end{aligned}$$

Fix $r_0 > 0$ such that

$$\mathcal{C} := \left\{x : |x| \leq r_0, \delta(x) \geq \frac{1}{2}|x-y|\right\} \subset \left\{x \in \Omega : \delta(x) \geq \frac{1}{2}|x-y|\right\}.$$

Then

$$\int_\Omega \delta(x)^{\alpha_+} u(x)^q dx \geq c'_{30} \int_{\mathcal{C}} |x-y|^{\alpha_+ - (N-1-\alpha_-)q} dx. \tag{4.35}$$

Since $q \geq q^*$, the integral on the right hand-side of (4.35) is divergent, which in turn implies that $u \notin L^q(\Omega; \delta^{\alpha_+})$. Thus we get a contradiction. □

Remark Interesting removability result in the supercritical case can be found in [8].

5 More general source

In this section, we assume that $(g \circ u)(x) = \delta(x)^\gamma \tilde{g}(u(x))$ where $\gamma > -1 - \alpha_+$ and $\tilde{g} : \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and continuous. Theorems H and I are obtained by using the method in [12].

5.1 Subcriticality

Let $\{g_n\}$ be a sequence of C^1 nonnegative functions defined on $\mathbb{R}_+$ such that

$$g_n(0) = \tilde{g}(0),\ \ g_n \leq g_{n+1} \leq \tilde{g},\ \ \sup_{\mathbb{R}_+} g_n = n \text{ and } \lim_{n\to\infty} \|g_n - \tilde{g}\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}_+)} = 0. \tag{5.1}$$

Put

$$\tilde{\gamma} := \min\{\alpha_+ + \gamma, -\alpha_-\} > -1. \tag{5.2}$$

In preparation for proving Theorem H, we establish the following lemma.

Lemma 5.1 *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$ and $\{g_n\} \subset C^1(\mathbb{R}_+)$ be a sequence satisfying (5.1). Assume (1.28) and (1.29) are satisfied. Then there exist $\bar{\lambda}, \theta_0 > 0$ and $\varrho_0 > 0$ depending on $\Lambda_0, \Lambda_1, N, \mu, \gamma$ and q_1 such that for every $\theta \in (0, \theta_0)$ and $\varrho \in (0, \varrho_0)$ the following problem*

$$-L_\mu v = \delta^\gamma g_n(v + \varrho \mathbb{K}_\mu[\nu]) \quad \text{in } \Omega, \qquad \operatorname{tr}^*(v) = 0 \tag{5.3}$$

admits a positive weak solution $v_n \in L_w^{q^*_{\tilde{\gamma}}}(\Omega; \delta^{\alpha_+ + \gamma}) \cap L^{q_1}(\Omega; \delta^{-\alpha_-})$ satisfying

$$\|v_n\|_{L_w^{q^*_{\tilde{\gamma}}}(\Omega;\delta^{\alpha_+ + \gamma})} + \|v_n\|_{L^{q_1}(\Omega;\delta^{\tilde{\gamma}})} \leq \bar{\lambda}. \tag{5.4}$$

Proof We shall use Schauder fixed point theorem to show the existence of a positive weak solution of (5.3). For $n \in \mathbb{N}$, define the operator $\mathbb{S}_n$ by

$$\mathbb{S}_n(v) := \mathbb{G}_\mu[\delta^\gamma g_n(v + \varrho\mathbb{K}_\mu[\nu])] \quad \forall v \in L^1_+(\Omega). \tag{5.5}$$

Set

$$\begin{aligned} &M_1(v) := \|v\|_{L_w^{q_\gamma^*}(\Omega;\delta^{\alpha_++\gamma})} \quad \forall v \in L_w^{q_\gamma^*}(\Omega;\delta^{\alpha_++\gamma}), \\ &M_2(v) := \|v\|_{L^{q_1}(\Omega;\delta^{\tilde{\gamma}})} \quad \forall v \in L^{q_1}(\Omega;\delta^{\tilde{\gamma}}), \\ &M(v) := M_1(v) + M_2(v) \quad \forall v \in L_w^{q_\gamma^*}(\Omega;\delta^{\alpha_++\gamma}) \cap L^{q_1}(\Omega;\delta^{\tilde{\gamma}}). \end{aligned} \tag{5.6}$$

Step 1: To estimate $L^1(\Omega;\delta^{\alpha_++\gamma})$-norm of $g_n(v + \varrho\mathbb{K}_\mu[\nu])$ for $v \in L_w^{q_\gamma^*}(\Omega;\delta^{\alpha_++\gamma}) \cap L^{q_1}(\Omega;\delta^{\tilde{\gamma}})$.

For $\lambda > 0$, set $A_\lambda := \{x \in \Omega : v + \varrho\mathbb{K}_\mu[\nu] > \lambda\}$ and $a(\lambda) := \int_{A_\lambda} \delta^{\alpha_++\gamma} dx$. We write

$$\begin{aligned} \left\|g_n(v + \varrho\mathbb{K}_\mu[\nu])\right\|_{L^1(\Omega;\delta^{\alpha_++\gamma})} &= \int_{A_1} g_n(v + \varrho\mathbb{K}_\mu[\nu])\delta^{\alpha_++\gamma} dx \\ &\quad + \int_{A_1^c} g_n(v + \varrho\mathbb{K}_\mu[\nu])\delta^{\alpha_++\gamma} dx \\ &=: I + II. \end{aligned} \tag{5.7}$$

We first estimate I from above. We see that

$$I = a(1)g_n(1) + \int_1^\infty a(s)dg_n(s).$$

Since (1.28) holds, it was proved in [12, Lemma 3.1] that there exists an increasing sequence of positive number $\{\ell_j\}$ such that

$$\lim_{j\to\infty} \ell_j = \infty \quad \text{and} \quad \lim_{j\to\infty} \ell_j^{-q_\gamma^*}\tilde{g}(\ell_j) = 0. \tag{5.8}$$

Consequently,

$$\lim_{j\to\infty} \ell_j^{-q_\gamma^*} g_n(\ell_j) = 0, \quad \forall n \in \mathbb{N}. \tag{5.9}$$

Observe that

$$\int_1^\infty a(s)dg_n(s) = \lim_{j\to\infty} \int_\lambda^{\ell_j} a(s)dg_n(s).$$

On the other hand, by (2.1) one gets, for every $s > 0$,

$$a(s) \le \left\|v + \varrho\mathbb{K}_\mu[\nu]\right\|^{q_\gamma^*}_{L_w^{q_\gamma^*}(\Omega;\delta^{\alpha_++\gamma})} s^{-q_\gamma^*} \le c_{31}(M_1(v) + c_{32}\varrho)^{q_\gamma^*} s^{-q_\gamma^*} \tag{5.10}$$

where $c_i = c_i(N, \mu, \gamma, \Omega)$ with $i = 31, 32$. Using (5.10), we obtain

$$
\begin{aligned}
&a(1)g_n(1) + \int_1^{\ell_j} a(s)dg_n(s)\\
&\quad\le c_{31}(M_1(v) + c_{32}\varrho)^{q_\gamma^*} g_n(1) + c_{31}(M_1(v) + c_{32}\varrho)^{q_\gamma^*} \int_1^{\ell_j} s^{-q_\gamma^*} dg_n(s)\\
&\quad\le c_{31}(M_1(v) + c_{32}\varrho)^{q_\gamma^*} \ell_j^{-q_\gamma^*} g_n(\ell_j) + c_{31}q_\gamma^*(M_1(v)\\
&\qquad + c_{32}\varrho)^{q_\gamma^*} \int_1^{\ell_j} s^{-1-q_\gamma^*} g_n(s)ds.
\end{aligned}
$$

By virtue of (5.8), letting $j \to \infty$ yields

$$
\begin{aligned}
I &\le c_{31}q_\gamma^*(M_1(v) + c_{32}\varrho)^{q_\gamma^*} \int_1^\infty s^{-1-q_\gamma^*} g_n(s)ds\\
&\le c_{33}\Lambda_0 M_1(v)^{q_\gamma^*} + c_{33}\Lambda_0\varrho^{q_\gamma^*}
\end{aligned}
\tag{5.11}
$$

where $c_{33} = c_{33}(N, \mu, \gamma, \Omega)$.

To handle the remaining term II, without lost of generality, we assume $q_1 \in (1, \frac{N+\tilde{\gamma}}{N-1-\alpha_-})$. Since $\tilde{g}$ satisfies condition (1.29) and $g_n \le \tilde{g}$, it follows that g_n satisfies this condition too. Hence

$$
\begin{aligned}
II &\le \Lambda_1 \int_{A_1^c} (v + \varrho\mathbb{K}_\mu[\nu])^{q_1}\delta^{\alpha_+ + \gamma}dx + \theta\int_{A_1^c}\delta^{\alpha_+ + \gamma}dx\\
&\le \Lambda_1 c_{34}\int_\Omega v^{q_1}\delta^{\alpha_+ + \gamma}dx + \Lambda_1 c_{34}\varrho^{q_1} + c_{34}\theta\\
&\le \Lambda_1 c_{35}M_2(v)^{q_1} + \Lambda_1 c_{34}\varrho^{q_1} + c_{34}\theta
\end{aligned}
\tag{5.12}
$$

where $c_i = c_i(N, \mu, q_1, \gamma, \Omega)$, $i = 34, 35$.

Combining (5.7), (5.11) and (5.12) yields

$$
\begin{aligned}
\left\|g_n(v + \varrho\mathbb{K}_\mu[\nu])\right\|_{L^1(\Omega;\delta^{\alpha_+ + \gamma})} \le\; & c_{33}\Lambda_0 M_1(v)^{q_\gamma^*} + c_{35}\Lambda_1 M_2(v)^{q_1}\\
&+ c_{34}\theta + d_\varrho
\end{aligned}
\tag{5.13}
$$

where $d_\varrho = c_{33}\Lambda_0\varrho^{q_\gamma^*} + c_{34}\Lambda_1\varrho^{q_1}$.

Step 2: To estimate M_1, M_2 and M.

From (2.4), we have

$$
\begin{aligned}
M_1(\mathbb{S}_n(v)) &= \left\|\mathbb{G}_\mu[\delta^\gamma g_n(v + \varrho\mathbb{K}_\mu[\nu])]\right\|_{L_w^{q_\gamma^*}(\Omega;\delta^{\alpha_+ + \gamma})}\\
&\le c_7\left\|g_n(v + \varrho\mathbb{K}_\mu[\nu])\right\|_{L^1(\Omega;\delta^{\alpha_+ + \gamma})}.
\end{aligned}
\tag{5.14}
$$

It follows that

$$
M_1(\mathbb{S}_n(v)) \le c_7c_{33}\Lambda_0 M_1(v)^{q_\gamma^*} + c_7c_{35}\Lambda_1 M_2(v)^{q_1} + c_7c_{34}\theta + c_7d_\varrho. \tag{5.15}
$$

Applying (2.4), we get

$$
\begin{aligned}
M_2(\mathbb{S}_n(v)) &= \left\|\mathbb{G}_\mu[\delta^\gamma g_n(v + \varrho\mathbb{K}_\mu[\nu])]\right\|_{L^{q_1}(\Omega;\delta^{\tilde{\gamma}})}\\
&\le c_{36}\left\|g_n(v + \varrho\mathbb{K}_\mu[\nu])\right\|_{L^1(\Omega;\delta^{\alpha_+ + \gamma})},
\end{aligned}
$$

which implies

$$
M_2(\mathbb{S}_n(v)) \le c_{36}c_{33}\Lambda_0 M_1(v)^{q_\gamma^*} + c_{36}c_{35}\Lambda_1 M_2(v)^{q_1} + c_{36}c_{34}\theta + c_{36}d_\varrho. \tag{5.16}
$$

Consequently,

$$M(\mathbb{S}_n(v)) \leq c_{37}\Lambda_0 M_1(v)^{q_\gamma^*} + c_{38}\Lambda_1 M_2(v)^{q_1} + c_{40}\theta + c_{39}d_\varrho \tag{5.17}$$

where $c_{37} = (c_7 + c_{36})c_{33}, c_{38} = (c_7 + c_{36})c_{35}, c_{39} = c_7 + c_{36}, c_{40} = (c_7 + c_{36})c_{34}$. Therefore if $M(\mathbb{S}_n(v)) \leq \lambda$ then

$$M(\mathbb{S}_n(v)) \leq c_{37}\Lambda_0\lambda^{q_\gamma^*} + c_{38}\Lambda_1\lambda^{q_1} + c_{40}\theta + c_{39}d_\varrho.$$

Since $q_\gamma^* > 1$ and $q_1 > 1$, there exist $\varrho_0 > 0$ and $\theta_0 > 0$ such that for any $\varrho \in (0, \varrho_0)$ and $\theta \in (0, \theta_0)$ the equation

$$c_{37}\Lambda_0\lambda^{q_\gamma^*} + c_{38}\Lambda_1\lambda^{q_1} + c_{40}\theta + c_{39}d_\varrho = \lambda$$

admits a largest root $\bar{\lambda} > 0$. Therefore,

$$M(v) \leq \bar{\lambda} \Longrightarrow M(\mathbb{S}_n(v)) \leq \bar{\lambda}. \tag{5.18}$$

Step 3: We apply Schauder fixed point theorem to our setting.

Set

$$\mathcal{O} := \{\phi \in L^1_+(\Omega) : M(\phi) \leq \bar{\lambda}\}.$$

Clearly, $\mathcal{O}$ is a convex subset of $L^1(\Omega)$. We shall show that $\mathcal{O}$ is a closed subset of $L^1(\Omega)$. Indeed, let $\{\phi_m\}$ be a sequence in $\mathcal{O}$ converging to ϕ in $L^1(\Omega)$. Obviously, $\phi \geq 0$. We can extract a subsequence, still denoted by $\{\phi_m\}$, such that $\phi_m \to \phi$ a.e. in Ω. Consequently, by Fatou's lemma, $M_i(\phi) \leq \liminf_{m\to\infty} M_i(\phi_m)$ for $i = 1, 2$. It follows that $M(\phi) \leq \bar{\lambda}$. So $\phi \in \mathcal{O}$ and therefore $\mathcal{O}$ is a closed subset of $L^1(\Omega)$.

In light of (5.13) and (5.18), $\mathbb{S}_n$ is well-defined on $\mathcal{O}$ and $\mathbb{S}_n(\mathcal{O}) \subset \mathcal{O}$.

We observe that $\mathbb{S}_n$ is continuous. Indeed, if $\phi_m \to \phi$ as $m \to \infty$ in $L^1(\Omega)$ then $g_n(\phi_m + \varrho\mathbb{K}_\mu[\nu]) \to g_n(\phi + \varrho\mathbb{K}_\mu[\nu])$ as $m \to \infty$ in $L^1(\Omega; \delta^{\alpha_+ +\gamma})$. By (2.4), $\mathbb{S}_n(\phi_m) \to \mathbb{S}_n(\phi)$ as $m \to \infty$ in $L^1(\Omega)$.

We next show that $\mathbb{S}_n$ is a compact operator. Let $\{\phi_m\} \subset \mathcal{O}$ and for each n put $\psi_m := \mathbb{S}_n(\phi_m)$. Hence $\{\Delta\psi_m\}$ is uniformly bounded in $L^p(G)$ for every compact subset $G \subset \Omega$. Therefore $\{\psi_m\}$ is uniformly bounded in $W^{1,p}(G)$. Consequently, there exists a subsequence, still denoted by $\{\psi_m\}$, and a function ψ such that $\psi_m \to \psi$ a.e. in Ω. By dominated convergence theorem, $\psi_m \to \psi$ in $L^1(\Omega)$. Thus $\mathbb{S}_n$ is compact.

By Schauder fixed point theorem there is a function $v_n \in L^1_+(\Omega)$ such that $\mathbb{S}_n(v_n) = v_n$ and $M(v_n) \leq \bar{\lambda}$ where $\bar{\lambda}$ is independent of n. Due to Proposition 2.4, $\mathrm{tr}^*(v_n) = 0$ and v_n is a nonnegative solution of (5.3). Moreover, there holds

$$-\int_\Omega v_n L_\mu\zeta dx = \int_\Omega \delta^\gamma g_n(v_n + \varrho\mathbb{K}_\mu[\nu])\zeta dx \quad \forall\zeta \in X(\Omega). \tag{5.19}$$

□

Proof of Theorem H Let $\theta \in (0, \theta_0)$ and $\varrho \in (0, \varrho_0)$. For each n, set $u_n = v_n + \varrho\mathbb{K}_\mu[\nu]$ where v_n is the solution constructed in Lemma 5.1. Then $\mathrm{tr}^*(u_n) = \varrho\nu$ and

$$-\int_\Omega u_n L_\mu\zeta dx = \int_\Omega \delta^\gamma g_n(u_n)\zeta dx - \varrho\int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta dx \quad \forall\zeta \in X(\Omega). \tag{5.20}$$

Since $\{v_n\} \subset \mathcal{O}$, the sequence $\{g_n(v_n + \varrho\mathbb{K}_\mu[\nu])\}$ is uniformly bounded in $L^1(\Omega; \delta^{\alpha_+ +\gamma})$ and the sequence $\{\frac{\mu}{\delta^2}v_n\}$ is uniformly bounded in $L^{q_1}(G)$ for every compact subset $G \subset \Omega$. As a consequence, $\{\Delta v_n\}$ is uniformly bounded in $L^1(\Omega)$. By regularity result for elliptic equations, there exists a subsequence, still denoted by $\{v_n\}$, and a function v such that $v_n \to v$ a.e. in Ω. Therefore $u_n \to u$ a.e. in Ω with $u = v + \varrho\mathbb{K}_\mu[\nu]$ and $g_n(u_n) \to \tilde{g}(u)$ a.e. in Ω.

We show that $u_n \to u$ in $L^1(\Omega; \delta^{-\alpha_-})$. Since $\{v_n\}$ is uniformly bounded in $L^{q_1}(\Omega; \delta^{\tilde{\gamma}})$, by (2.5), we derive that $\{u_n\}$ is uniformly bounded in $L^{q_1}(\Omega; \delta^{-\alpha_-})$. Due to Hölder inequality, $\{u_n\}$ is uniformly integrable with respect to $\delta^{-\alpha_-}dx$. We invoke Vitali's convergence theorem to derive that $u_n \to u$ in $L^1(\Omega; \delta^{-\alpha_-})$.

We next prove that $g_n(u_n) \to \tilde{g}(u)$ in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$. For $\lambda > 0$ and $n \in \mathbb{N}$ set $B_{n,\lambda} := \{x \in \Omega : u_n > \lambda\}$ and $b_n(\lambda) := \int_{B_{n,\lambda}} \delta^{\alpha_+ + \gamma} dx$. For any Borel set $E \subset \Omega$,

$$\begin{aligned}\int_E g_n(u_n)\delta^{\alpha_+ + \gamma}dx &= \int_{E\cap B_{n,\lambda}} g_n(u_n)\delta^{\alpha_+ + \gamma}dx + \int_{E\cap B_{n,\lambda}^c} g_n(u_n)\delta^{\alpha_+ + \gamma}dx \\ &\le \int_{B_{n,\lambda}} g_n(u_n)\delta^{\alpha_+ + \gamma}dx + \Theta_\lambda \int_E \delta^{\alpha_+ + \gamma}dx \\ &\le b_n(\lambda)g_n(\lambda) + \int_\lambda^\infty b_n(s)dg_n(s) + \Theta_\lambda \int_E \delta^{\alpha_+ + \gamma}dx. \qquad (5.21)\end{aligned}$$

where $\Theta_\lambda := \sup_{[0,\lambda]} g$. By proceeding as in the proof of Lemma 5.1, we deduce

$$\begin{aligned}b_n(\lambda)g_n(\lambda) + \int_\lambda^\infty b_n(s)dg_n(s) &\le c_{41} \textstyle\int_\lambda^\infty s^{-1-q_\gamma^*} g_n(s)ds \\ &\le c_{41} \textstyle\int_\lambda^\infty s^{-1-q_\gamma^*} \tilde{g}(s)ds \qquad (5.22)\end{aligned}$$

where c_{41} depends on N, μ, γ and Ω. Note that the term on the right hand-side of (5.22) tends to 0 as $\lambda \to \infty$. Therefore for any $\varepsilon > 0$, there exists $\lambda > 0$ such that the right hand-side of (5.22) is smaller than $\frac{\varepsilon}{2}$. Fix such λ and put $\eta = \frac{\varepsilon}{2\Theta_\lambda}$. Then, by (5.21),

$$\int_E \delta(x)^{\alpha_+ + \gamma}dx \le \eta \Longrightarrow \int_E g_n(u_n)\delta(x)^{\alpha_+ + \gamma}dx < \varepsilon.$$

Therefore the sequence $\{g_n(u_n)\}$ is uniformly integrable with respect to $\delta^{\alpha_+ + \gamma}dx$. Due to Vitali convergence theorem, we deduce that $g_n(u_n) \to \tilde{g}(u)$ in $L^1(\Omega; \delta^{\alpha_+ + \gamma})$.

Finally, by sending $n \to \infty$ in each term of (5.20) we obtain

$$-\int_\Omega uL_\mu\zeta dx = \int_\Omega \delta^\gamma \tilde{g}(u)\zeta dx - \varrho \int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta dx \quad \forall \zeta \in X(\Omega). \qquad (5.23)$$

By Theorem A, u is a nonnegative weak solution of (1.27). □

5.2 Sublinearity

In this subsection we deal with the case where g is sublinear.

Lemma 5.2 *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$ and $\{g_n\} \subset C^1(\mathbb{R}_+)$ be a sequence satisfying (5.1). Assume (1.30) is satisfied. Then for every $\varrho > 0$ problem (5.3) admits a nonnegative solution v_n satisfying*

$$\|v_n\|_{L^1(\Omega;\delta^{\tilde{\gamma}})} \le \tilde{\lambda} \qquad (5.24)$$

where $\tilde{\gamma}$ is as in (5.2) and $\tilde{\lambda}$ depends on Λ_2, q_2, N, μ and γ.

Proof The proof is similar to that of Lemma 5.1, also based on Schauder fixed point theorem. So we point out only the main modifications. Let $\mathbb{S}_n$ be the operator defined in (5.5). Fix $q_3 \in \left(1, \frac{N+\tilde{\gamma}}{N-1-\alpha_-}\right)$ and put

$$N(v) := \|v\|_{L^{q_3}(\Omega;\delta^{\tilde{\gamma}})} \quad \forall v \in L^{q_3}(\Omega;\delta^{\tilde{\gamma}}).$$

Combining (2.4), (2.5) and (1.30) leads to

$$\begin{aligned}
N(\mathbb{S}_n(v)) &\le c_{42} \left\| g_n(v + \varrho \mathbb{K}_\mu[\nu]) \right\|_{L^1(\Omega;\delta^{\alpha_+ + \gamma})} \\
&\le c_{42} \int_\Omega \Lambda_2 (v + \varrho \mathbb{K}_\mu[\nu])^{q_2} \delta^{\alpha_+ + \gamma} dx + c_{42}\theta \int_\Omega \delta^{\alpha_+ + \gamma} dx \\
&\le c_{42}\Lambda_2 \int_\Omega v^{q_2} \delta^{\alpha_+ + \gamma} dx + c_{43}(\varrho^{q_2} + \theta) \\
&\le c_{44}\Lambda_2 \left(\int_\Omega v^{q_3} \delta^{\alpha_+ + \gamma} dx \right)^{\frac{q_2}{q_3}} + c_{43}(\varrho^{q_2} + \theta) \\
&\le c_{44}\Lambda_2 N(v)^{q_2} + c_{43}(\varrho^{q_2} + \theta)
\end{aligned}$$

where $c_i = c_i(N, \mu, \gamma, \Omega, q_2)$ $(42 \le i \le 44)$. Therefore, if $N(v) \le \lambda$ for some $\lambda > 0$ then

$$N(\mathbb{S}_n(v)) \le c_{44}\Lambda_2 \lambda^{q_2} + c_{43}(\varrho^{q_2} + \theta).$$

Consider the following algebraic equation

$$c_{44}\Lambda_2 \lambda^{q_2} + c_{43}(\varrho^{q_2} + \theta) = \lambda. \tag{5.25}$$

If $q_2 < 1$ then for any $\varrho > 0$ (5.25) admits a unique positive root $\tilde{\lambda}$. If $q_2 = 1$ then for Λ_2 small such that $c_{44}\Lambda_2 < 1$ and $\varrho > 0$ Eq. (5.25) admits a unique positive root $\tilde{\lambda}$. Therefore,

$$N(v) \le \tilde{\lambda} \Longrightarrow N(\mathbb{S}_n(v)) \le \tilde{\lambda}. \tag{5.26}$$

By proceeding as in the proof of Lemma 5.1, one can prove that $\mathbb{S}_n$ is a continuous, compact operator from the closed, convex set

$$\tilde{\mathcal{O}} := \{v \in L^1_+(\Omega) : N(v) \le \tilde{\lambda}\}$$

into itself. Thus by appealing to Schauder fixed point theorem, we see that there exists a function $v_n \in L^1_+(\Omega)$ such that $\mathbb{S}_n(v_n) = v_n$ and $N(v_n) \le \tilde{\lambda}$ with $\tilde{\lambda}$ being independent of n. By Proposition 2.4, $\mathrm{tr}^*(v_n) = 0$ and v_n is a nonnegative solution of (5.3). Moreover (5.19) holds. □

Proof of Theorem I Let v_n be the solution of (5.3) constructed in Lemma 5.2. Put $u_n = v_n + \varrho \mathbb{K}_\mu[\nu]$ then u_n satisfies (5.20). By a similar argument as in the proof of Theorem H, there exists a subsequence, still denoted by $\{u_n\}$ and a function u such that $u_n \to u$ a.e. in Ω. Since $\{v_n\} \subset \tilde{\mathcal{O}}$, it follows that $\{v_n\}$ is uniformly bounded in $L^{q_3}(\Omega;\delta^{-\alpha_-})$, so is $\{u_n\}$. By Holder inequality, $\{u_n\}$ is uniformly integrable in $L^1(\Omega;\delta^{-\alpha_-})$. Due to (1.30), $\{g_n(u_n)\}$ is uniformly integrable in $L^1(\Omega;\delta^{\alpha_+ + \gamma})$. Vitali convergence theorem implies that $u_n \to u$ in $L^1(\Omega;\delta^{-\alpha_-})$ and $g_n(u_n) \to \tilde{g}(u)$ in $L^1(\Omega;\delta^{\alpha_+ + \gamma})$. Letting $n \to \infty$ in (5.20), we conclude that u is a positive solution of (1.27). □

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References

1. Ancona, A.: Theorié du potentiel sur les graphes et les variétés. In Ecole d'été de Probabilités de Saint-Flour XVIII-1988. Springer Lecture Notes in Mathematics, vol. 1427, pp. 1–112 (1990)
2. Ancona, A.: Negatively curved manifolds, elliptic operators and the Martin boundary. Ann. Math. (2) **125**, 495–536 (1987)
3. Ancona, A., Marcus, M.: Positive solutions of a class of semilinear equations with absorption and schrodinger equations. J. Math. Pures et Appl. **104**, 587–618 (2015)
4. Bandle, C., Moroz, V., Reichel, W.: Boundary blowup type sub-solutions to semilinear elliptic equations with Hardy potential. J. Lond. Math. Soc. **2**, 503–523 (2008)
5. Bandle, C., Moroz, V., Reichel, W.: Large solutions to semilinear elliptic equations with Hardy potential and exponential nonlinearity. In: Around the research of Vladimir Maz'ya. II, 1-22, International Mathematical Series (New York), vol. 12, Springer, New York (2010)
6. Bandle, C., Marcus, M., Moroz, V.: Boundary singularities of solutions of semilinear elliptic equations in the half-space with a Hardy potential. arXiv:1604.08830
7. Bidaut-Véron, M.F., Vivier, L.: An elliptic semilinear equation with source term involving boundary measures: the subcritical case. Rev. Mat. Iberoamericana **16**, 477–513 (2000)
8. Bidaut-Véron, M.F., Hoang, G., Nguyen, Q.H., Véron, L.: An elliptic semilinear equation with source term and boundary measure data: the supercritical case. J. Funct. Anal. **269**, 1995–2017 (2015)
9. Bidaut-Véron, M.F., Yarur, C.: Semilinear elliptic equations and systems with measure data: existence and a priori estimates. Adv. Differ. Equ. **7**, 257–296 (2002)
10. Brezis, H., Cabré, X.: Some simple nonlinear PDE's without solutions. Boll. Unione Mat. Italiana **8**, 223–262 (1998)
11. Brezis, H., Marcus, M.: Hardy inequalities revisited. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **25**, 217–237 (1997)
12. Chen, H., Felmer, P., Véron, L.: *Elliptic equations involving general subcritical source nonlinearity and measures*. arxiv:1409.3067 (2014)
13. Dávila, J., Dupaigne, L.: Hardy-type inequalities. J. Eur. Math. Soc. **6**, 335–365 (2004)
14. Filippas, S., Moschini, L., Tertikas, A.: Sharp two-sided heat kernel estimates for critical Schrodinger operators on bounded domains. Commun. Math. Phys. **273**, 237–281 (2007)
15. Gkikas, K.T., Véron, L.: Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials. Nonlinear Anal. **121**, 469–540 (2015)
16. Gkikas, K.T., Nguyen, P.T.: On the existence of weak solutions of semilinear elliptic equations and systems with Hardy potential. arXiv:1609.06671
17. Kalton, N., Verbitsky, I.: Nonlinear and weighted norm inequalities. Trans. Am. Math. Soc. **351**, 3441–3497 (1999)
18. Marcus, M.: Complete classification of the positive solutions of $-\Delta u + u^q = 0$. Jl. d'Anal. Math. **117**, 187–220 (2012)
19. Marcus, M., Moroz,V.: Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential. arXiv:1603.09265
20. Marcus, M., Nguyen, P.T.: Moderate solutions of semilinear elliptic equations with Hardy potential. Ann. Inst. H. Poincaré Anal. Non Linéaire **34**, 69–88 (2017)
21. Marcus, M., Mizel, V.J., Pinchover, Y.: On the best constant for Hardy's inequality in $\mathbb{R}^N$. Trans. Am. Math. Soc. **350**, 3237–3255 (1998)
22. Marcus, M., Shafrir, I.: An eigenvalue problem related to Hardy's L^p inequality. Ann. Scuola. Norm. Sup. Pisa. Cl. Sci. **29**, 581–604 (2000)
23. Marcus, M., Véron, L.: Nonlinear second order elliptic equations involving measures. In: De Gruyter Series in Nonlinear Analysis and Applications, vol. 21. De Gruyter, Berlin (2014)
24. Pinchover, Y., Tintarev, K.: A ground state alternative for singular Schrödinger operators. J. Funct. Anal. **230**, 65–77 (2006)
25. Véron, L., Yarur, C.: Boundary value problems with measures for elliptic equations with singular potentials. J. Funct. Anal. **262**, 733–772 (2012)

CHAPTER 4

Semilinear elliptic equations and systems with Hardy potentials

This chapter, which is based on a collaboration with Gkikas [78], is a continuation of our study on semilinear equations with a Hardy potential. We offer a unified approach and go further in the analysis of the boundary value problems with both interior and boundary measure data. We also extend several existence results for semilinear equations to systems.

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On the existence of weak solutions of semilinear elliptic equations and systems with Hardy potentials

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Abstract

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded C^2 domain and $\delta(x) = \mathrm{dist}\,(x, \partial\Omega)$. Put $L_\mu = \Delta + \frac{\mu}{\delta^2}$ with $\mu > 0$. In this paper, we provide various necessary and sufficient conditions for the existence of weak solutions to

$$-L_\mu u = u^p + \tau \quad \text{in } \Omega, \qquad u = \nu \quad \text{on } \partial\Omega,$$

where $\mu > 0$, $p > 0$, τ and ν are measures on Ω and $\partial\Omega$ respectively. We then establish existence results for the system

$$\begin{cases} -L_\mu u = \epsilon\, v^p + \tau & \text{in } \Omega, \\ -L_\mu v = \epsilon\, u^{\tilde{p}} + \tilde{\tau} & \text{in } \Omega, \\ u = \nu, \quad v = \tilde{\nu} & \text{on } \partial\Omega, \end{cases}$$

where $\epsilon = \pm 1$, $p > 0$, $\tilde{p} > 0$, τ and $\tilde{\tau}$ are measures on Ω, ν and $\tilde{\nu}$ are measures on $\partial\Omega$. We also deal with elliptic systems where the nonlinearities are more general.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a bounded C^2 domain, $\delta(x) = \mathrm{dist}\,(x, \partial\Omega)$ and $g \in C(\mathbb{R})$. Put $L_\mu := \Delta + \frac{\mu}{\delta^2}$. In the present paper we study semilinear problems with *Hardy potential* of the form

$$-L_\mu u = g(u) + \tau \quad \text{in } \Omega, \tag{1.1}$$

where $\mu > 0$, τ is a Radon measure on Ω.

The boundary value problem with measures for (1.1) without Hardy potential and with power absorption nonlinearity, i.e. $\mu = 0$, $\tau = 0$, $g(u) = -|u|^{p-1}u$, $p > 1$, was well understood in the literature, starting with a work by Gmira and Véron [10]. It was proved that there is the critical exponent $p^* := \frac{N+1}{N-1}$ in the sense that if $p \in (1, p^*)$ then there is a unique weak solution for every finite measure ν on Ω, while if $p \in [p^*, \infty)$ there exists no solution with a boundary isolated singularity. Marcus and Véron [15,16] studied this problem by introducing a notion of *boundary trace*, providing a complete description of isolated singularities in the *subcritical case*, i.e. $1 < p < p^*$, and giving a removability result in the *supercritical case*, i.e. $p \geq p^*$.

The solvability for boundary value problem for (1.1) without Hardy potential and with *power source term*, namely $\mu = 0$, $\tau = 0$, $g(u) = u^p$, $p > 1$, was studied by Bidaut-Véron and Vivier [4] in connection with sharp estimates of the Green operator and the Poisson operator associated to $(-\Delta)$ in Ω. They proved that, in the subcritical case $1 < p < p^*$, the problem admits a solution if and only if the total mass of the boundary datum ν is sufficiently small. Afterwards, Bidaut-Véron and Yarur [6] reconsidered this type of problem in a more general setting and provided a necessary and sufficient condition for the existence of solutions. Recently, Bidaut-Véron et al. [5] provided new criteria for the existence of solutions with $p > 1$ in terms of the capacity associated to the Besov spaces.

Let $\phi \geq 0$ in Ω and $p \geq 1$, we denote by $L^p(\Omega; \phi)$ the space of all function v on Ω satisfying $\int_\Omega |v|^p \phi dx < \infty$. We denote by $\mathfrak{M}(\Omega; \phi)$ the space of Radon measures τ on Ω satisfying $\int_\Omega \phi \, d|\tau| < \infty$ and by $\mathfrak{M}^+(\Omega; \phi)$ the nonnegative cone of $\mathfrak{M}(\Omega; \phi)$. When $\phi \equiv 1$, we use the notations $\mathfrak{M}(\Omega)$ and $\mathfrak{M}^+(\Omega)$. We also denote by $\mathfrak{M}(\partial\Omega)$ the space of finite measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the nonnegative cone of $\mathfrak{M}(\partial\Omega)$.

Let G_μ and K_μ be the Green kernel and Martin kernel of $-L_\mu$ in Ω, $\mathbb{G}_\mu$ and $\mathbb{K}_\mu$ be the corresponding Green operator and Martin operator (see [14,9] for more details). Let C_H be Hardy constant, namely

$$C_H := \inf_{v \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla v|^2 dx}{\int_\Omega (v/\delta)^2 dx} \tag{1.2}$$

then it is well known that $0 < C_H \leq \frac{1}{4}$ and if Ω is convex then $C_H = \frac{1}{4}$ (see for example [12]). Moreover the infimum is achieved if and only if $C_H < \frac{1}{4}$. When $-\Delta\delta \geq 0$ in Ω in the sense of distributions, the first eigenvalue λ_μ of L_μ in Ω is positive, i.e.

$$\lambda_\mu := \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla\varphi|^2 - \frac{\mu}{\delta^2}\varphi^2) dx}{\int_\Omega \varphi^2 dx} > 0. \tag{1.3}$$

For $\mu \in (0, \frac{1}{4}]$, denote by α the following *fundamental exponent*

$$\alpha := \frac{1}{2}(1 + \sqrt{1 - 4\mu}). \tag{1.4}$$

Notice that $\frac{1}{2} < \alpha < 1$. The eigenfunction φ_μ associated to λ_μ with the normalization $\int_\Omega (\varphi_\mu/\delta)^2 dx = 1$ satisfies $c^{-1}\delta^\alpha \leq \varphi_\mu \leq c\delta^\alpha$ for some constant $c > 0$ (see [7]).

In relation to Hardy constant, Bandle et al. [3] classified large solutions of the linear equation

$$-L_\mu u = 0 \quad \text{in } \Omega, \tag{1.5}$$

and of the associated nonlinear equation with power absorption

$$-L_\mu u + u^p = 0 \quad \text{in } \Omega. \tag{1.6}$$

In [14], Marcus and P.-T. Nguyen studied boundary value problem for (1.5) and (1.6) with $\mu \in (0, C_H)$ in measure framework by introducing a notion of *normalized boundary trace* which is defined as follows:

Definition 1.1. A function $u \in L^1_{loc}(\Omega)$ possesses a *normalized boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that

$$\lim_{\beta \to 0} \beta^{\alpha-1} \int_{\{x \in \Omega : \delta(x) = \beta\}} |u - \mathbb{K}_\mu[\nu]| dS = 0. \tag{1.7}$$

The normalized boundary trace is denoted by $\mathrm{tr}^*(u)$.

The restriction $\mu \in (0, C_H)$ in [14] is due to the fact that in this case L_μ is *weakly coercive* in $H_0^1(\Omega)$ and consequently by a result of Ancona [2, Remark p. 523] there is a $(1-1)$ correspondence between $\mathfrak{M}^+(\partial\Omega)$ and the class of positive L_μ harmonic functions, namely any positive L_μ harmonic function u can be written in a unique way under the form $u = \mathbb{K}_\mu[\nu]$ for some $\nu \in \mathfrak{M}^+(\partial\Omega)$.

The notion of normalized boundary trace was proved [14] to be an appropriate generalization of the classical boundary trace to the setting of Hardy potentials, giving a characterization of *moderate solutions* of (1.6). In addition, it was showed in [14] that there exists the critical exponent

$$p_\mu := \frac{N+\alpha}{N+\alpha-2} \tag{1.8}$$

such that if $p \in (1, p_\mu)$ then there exists a unique solution of (1.6) with $\operatorname{tr}^*(u) = \nu$ for every finite measure ν on $\partial\Omega$, while if $p \geq p_\mu$ there is no solution of (1.6) with an isolated boundary singularity. Marcus and Moroz [13] then *extended the notion of normalized boundary trace to the case* $\mu < \frac{1}{4}$ and employed it to investigate (1.6). When $\mu = \frac{1}{4}$, L_μ is no longer weakly coercive and hence Ancona's result cannot be applied. However, Gkikas and Véron [9] observed that *if the first eigenvalue of* $-L_{\frac{1}{4}}$ *is positive* then the kernel $K_{\frac{1}{4}}(\cdot, y)$ with pole at $y \in \partial\Omega$ is unique up to a multiplication and any positive $L_{\frac{1}{4}}$ harmonic function u admits such a representation. Based on that observation, they considered the boundary value problem with measures for (1.6), fully classifying isolated singularities in the subcritical case $p \in (1, p_\mu)$ and providing removability result in the supercritical case $p \geq p_\mu$. A main ingredient in [9] is the notion of boundary trace which is defined in a *dynamic way* and is recalled below.

Let $D \Subset \Omega$ and $x_0 \in D$. If $h \in C(\partial D)$ then the following problem

$$\begin{cases} -L_\mu u = 0 & \text{in } D, \\ u = h & \text{on } \partial D, \end{cases} \tag{1.9}$$

admits a unique solution which allows to define the L_μ-harmonic measure $\omega_D^{x_0}$ on ∂D by

$$u(x_0) = \int_{\partial D} h(y) d\omega_D^{x_0}(y). \tag{1.10}$$

A sequence of domains $\{\Omega_n\}$ is called a *smooth exhaustion* of Ω if $\partial\Omega_n \in C^2$, $\overline{\Omega_n} \subset \Omega_{n+1}$, $\cup_n \Omega_n = \Omega$ and $\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega)$. For each n, let $\omega_{\Omega_n}^{x_0}$ be the $L_\mu^{\Omega_n}$-harmonic measure on $\partial\Omega_n$.

Definition 1.2. A function u possesses a *boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ such that for any smooth exhaustion $\{\Omega_n\}$ of Ω,

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \zeta u \, d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega} \zeta \, d\nu \quad \forall \zeta \in C(\overline{\Omega}). \tag{1.11}$$

The boundary trace of u is denoted by $\operatorname{tr}(u)$.

It is worthy mentioning that in Definition 1.2, μ is allowed to belong to the range $(0, \frac{1}{4}]$ provided $\lambda_\mu > 0$.

In parallel, semilinear equations with Hardy potential and source term

$$-L_\mu u = u^p \quad \text{in } \Omega \tag{1.12}$$

were treated by Bidaut-Véron et al. [5] and by P.-T. Nguyen [18] and a fairly complete description of the profile of solutions to (1.12) was obtained in subcritical case $p < p_\mu$ (see [18]) and in supercritical case $p \geq p_\mu$ (see [5]).

Our first contribution is to show that *the notion of normalized boundary trace given in Definition 1.1 is equivalent to that in Definition 1.2 by examining* $\operatorname{tr}(\mathbb{G}_\mu^\Omega[\tau]) = \operatorname{tr}^*(\mathbb{G}_\mu^\Omega[\tau])$ *and* $\operatorname{tr}(\mathbb{K}_\mu^\Omega[\nu]) = \operatorname{tr}^*(\mathbb{K}_\mu^\Omega[\nu])$. This enables to establish important results for the boundary value problem for linear equations (see Proposition 2.13) which in turn forms a basic to study the boundary value problem for

$$\begin{cases} -L_\mu u = g(u) + \tau & \text{in } \Omega, \\ \operatorname{tr}(u) = \nu. \end{cases} \tag{1.13}$$

When dealing with (1.13), one encounters the following difficulties. The first one is due to the presence of the Hardy potential in the linear part of the equations. More precisely, since the singularity of the potential at the boundary is too strong, some important tools such as Hopf's lemma, the classical notion of boundary trace, are invalid, and therefore the system cannot be handled via classical elliptic PDEs methods. The second one comes from the interplay between the nonlinearity, the Hardy potential and measure data. The interaction between the difficulties generates an intricate dynamics both in Ω and near $\partial\Omega$ and leads to disclose new type of results.

Convention. Throughout the paper, unless otherwise stated, we assume that $\mu \in (0, \frac{1}{4}]$ *and the first eigenvalue* λ_μ *of* $-L_\mu$ *in* Ω *is positive*. We emphasize that if $\mu \in (0, C_H)$ then $\lambda_\mu > 0$.

Definition 1.3. (i) The space of test functions is defined as

$$\mathbf{X}_\mu(\Omega) := \{\zeta \in H^1_{loc}(\Omega) : \delta^{-\alpha}\zeta \in H^1(\Omega, \delta^{2\alpha}),\ \delta^{-\alpha} L_\mu \zeta \in L^\infty(\Omega)\}. \tag{1.14}$$

(ii) Let $(\tau, \nu) \in \mathfrak{M}(\Omega, \delta^\alpha) \times \mathfrak{M}(\partial\Omega)$. We say that u is a *weak solution* of (1.13) if $u \in L^1(\Omega; \delta^\alpha)$, $g(u) \in L^1(\Omega; \delta^\alpha)$ and

$$-\int_\Omega u L_\mu \zeta \, dx = \int_\Omega g(u)\zeta \, dx + \int_\Omega \zeta \, d\tau - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta \, dx \quad \forall \zeta \in \mathbf{X}_\mu(\Omega). \tag{1.15}$$

Main properties of solutions of (1.13) are established in the following proposition.

Proposition A. *Let* $\tau \in \mathfrak{M}(\Omega; \delta^\alpha)$ *and* $\nu \in \mathfrak{M}(\partial\Omega)$. *The following statements are equivalent.*

(i) *u is a weak solution of* (1.13).

(ii) $g(u) \in L^1(\Omega; \delta^\alpha)$ *and*

$$u = \mathbb{G}_\mu[g(u)] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu]. \tag{1.16}$$

(iii) $u \in L^1_{loc}(\Omega)$, $g(u) \in L^1_{loc}(\Omega)$, *$u$ is a distributional solution of* (1.1) *and* $\operatorname{tr}(u) = \nu$.

This allows to establish necessary and sufficient conditions for the existence of a weak solution of

$$\begin{cases} -L_\mu u = u^p + \sigma\tau & \text{in } \Omega, \\ \operatorname{tr}(u) = \varrho\nu. \end{cases} \tag{1.17}$$

Theorem B. *Let $\tau \in \mathfrak{M}^+(\Omega;\delta^\alpha)$, $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $p > 0$.*

(i) *Assume $0 < p < p_\mu$. Then there exists a constant $C > 0$ such that*

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^p] \le C\mathbb{K}_\mu[\nu] \quad \textit{a.e. in } \Omega. \tag{1.18}$$

(ii) *Assume $0 < p < p_\mu$. Then there exists a constant $C > 0$ such that*

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] \le C\mathbb{G}_\mu[\tau] \quad \textit{a.e. in } \Omega. \tag{1.19}$$

(iii) *If (1.18) and (1.19) hold then problem (1.17) admits a weak solution u satisfying*

$$\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] \le u \le C(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu]) \textit{ a.e. in } \Omega \tag{1.20}$$

for $\sigma > 0$ and $\varrho > 0$ small enough if $p > 1$, for any $\sigma > 0$ and $\varrho > 0$ if $0 < p < 1$.

(iv) *If $p > 1$ and (1.17) admits a weak solution then (1.18) and (1.19) hold with constant $C = \frac{1}{p-1}$.*

(v) *Assume $0 < p < p_\mu$. Then there exists a constant $C > 0$ such that for any weak solution u of (1.17) there holds*

$$\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] \le u \le C(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] + \delta^\alpha) \textit{ a.e. in } \Omega. \tag{1.21}$$

In order to study (1.17) in the supercritical case, i.e. $p \ge p_\mu$, we make use of the capacities introduced in [5] which is recalled below. For $0 \le \theta \le \beta < N$, set

$$N_{\theta,\beta}(x,y) := \frac{1}{|x-y|^{N-\beta}\max\{|x-y|,\delta(x),\delta(y)\}^\theta} \quad \forall (x,y) \in \overline{\Omega}\times\overline{\Omega},\ x \ne y, \tag{1.22}$$

$$\mathbb{N}_{\theta,\beta}[\tau](x) := \int_{\overline{\Omega}} N_{\theta,\beta}(x,y)d\tau \quad \forall \tau \in \mathfrak{M}^+(\overline{\Omega}). \tag{1.23}$$

For $a > -1$, $0 \le \theta \le \beta < N$ and $s > 1$, define $\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}$ by

$$\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E) := \inf\left\{ \int_{\overline{\Omega}} \delta^a\phi^s\,dx : \ \phi \ge 0,\ \mathbb{N}_{\theta,\beta}[\delta^a\phi] \ge \chi_E \right\}, \tag{1.24}$$

for any Borel set $E \subset \overline{\Omega}$. For $\theta \in (0, N-1)$ and $s > 0$, let $\mathrm{Cap}^{\partial\Omega}_{\theta,s}$ be the capacity defined in [5, Definition 1.1]. Notice that if $\theta s > N-1$ then $\mathrm{Cap}^{\partial\Omega}_{\theta,s}(\{z\}) > 0$ for every $z \in \partial\Omega$.

Theorem C. *Let $\tau \in \mathfrak{M}^+(\Omega;\delta^\alpha)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. Assume $p>1$. Then the following statements are equivalent.*

(i) *There exists $C>0$ such that the following inequalities hold*

$$\int_E \delta^\alpha d\tau \le C\,Cap^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2},p'}(E) \quad \forall \text{ Borel } E \subset \overline{\Omega}, \tag{1.25}$$

$$\nu(F) \le C\,Cap^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}(F) \quad \forall \text{ Borel } F \subset \partial\Omega. \tag{1.26}$$

(ii) *There exists a positive constant C such that* (1.18) *and* (1.19) *hold.*

(iii) *Problem* (1.17) *has a positive weak solution for $\sigma>0$ and $\varrho>0$ small enough.*

Remark. When $\tau=0$, Theorem C covers Theorem B (i), (iii) due to the fact that $\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}(\{z\})>c>0$ for every $z\in\partial\Omega$ if $1<p<p_\mu$. Also if $1<p<p_\mu$ then (see Lemma 3.10)

$$\inf_{\xi\in\Omega} \mathrm{Cap}^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2},p'}(\{\xi\})>0,$$

which implies the statements (ii) and (iii) in Theorem B.

The next goal of the present paper is the study of weak solutions of semilinear elliptic system involving Hardy potential

$$\begin{cases} -L_\mu u = g(v)+\tau & \text{in } \Omega,\\ -L_\mu v = \tilde{g}(u)+\tilde{\tau} & \text{in } \Omega,\\ \operatorname{tr}(u)=\nu, \quad \operatorname{tr}(v)=\tilde{\nu} \end{cases} \tag{1.27}$$

where $\tau,\ \tilde{\tau}\in\mathfrak{M}(\Omega;\delta^\alpha)$, $\nu,\tilde{\nu}\in\mathfrak{M}(\partial\Omega)$, $g,\tilde{g}\in C(\mathbb{R})$.

Definition 1.4. A pair (u,v) is called a weak solution of (1.27) if $u\in L^1(\Omega;\delta^\alpha)$, $v\in L^1(\Omega;\delta^\alpha)$, $\tilde{g}(u)\in L^1(\Omega;\delta^\alpha)$, $g(v)\in L^1(\Omega;\delta^\alpha)$ and

$$\begin{aligned} -\int_\Omega uL_\mu\zeta\,dx &= \int_\Omega g(v)\zeta\,dx + \int_\Omega \zeta\,d\tau - \int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta\,dx,\\ -\int_\Omega vL_\mu\zeta\,dx &= \int_\Omega \tilde{g}(u)\zeta\,dx + \int_\Omega \zeta\,d\tilde{\tau} - \int_\Omega \mathbb{K}_\mu[\tilde{\nu}]L_\mu\zeta\,dx \quad \forall\zeta\in\mathbf{X}_\mu(\Omega). \end{aligned} \tag{1.28}$$

A counterpart of Proposition A in the case of systems is the following:

Proposition D. *Let $\tau,\tilde{\tau}\in\mathfrak{M}(\Omega;\delta^\alpha)$ and $\nu,\tilde{\nu}\in\mathfrak{M}(\partial\Omega)$. Then the following statements are equivalent.*

(i) (u,v) *is a weak solution of* (1.27).
(ii) $\tilde g(u) \in L^1(\Omega;\delta^\alpha)$, $g(v) \in L^1(\Omega;\delta^\alpha)$ *and*

$$u = \mathbb{G}_\mu[g(v)] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu], \quad v = \mathbb{G}_\mu[\tilde g(u)] + \mathbb{G}_\mu[\tilde\tau] + \mathbb{K}_\mu[\tilde\nu]. \tag{1.29}$$

(iii) $(u,v) \in (L^1_{loc}(\Omega))^2$, $(g(v),\tilde g(u)) \in (L^1_{loc}(\Omega))^2$, (u,v) *is a solution of*

$$\begin{cases} -L_\mu u = g(v) + \tau & \text{in } \Omega, \\ -L_\mu v = \tilde g(u) + \tilde\tau & \text{in } \Omega, \end{cases} \tag{1.30}$$

in the sense of distributions and $\operatorname{tr}(u) = \nu$ *and* $\operatorname{tr}(v) = \tilde\nu$.

Elliptic systems arise in biological applications (e.g. population dynamics) or physical applications (e.g. models of nuclear reactor) and have been drawn a lot of attention (see [8,19] and references therein). A typical case is Lane–Emden system, i.e. system (1.27) with $\mu = 0$, $g(v) = v^p$, $\tilde g(u) = u^{\tilde p}$. Bidaut-Véron and Yarur [6] proved various existence results for Lane–Emden system under conditions involving the following exponents

$$q := \tilde p \frac{p+1}{\tilde p + 1}, \quad \tilde q := p \frac{\tilde p + 1}{p+1}. \tag{1.31}$$

We first treat the system

$$\begin{cases} -L_\mu u = v^p + \sigma\tau & \text{in } \Omega, \\ -L_\mu v = \tilde u^{\tilde p} + \tilde\sigma\tilde\tau & \text{in } \Omega, \\ \operatorname{tr}(u) = \varrho\nu, \quad \operatorname{tr}(v) = \tilde\varrho\tilde\nu, \end{cases} \tag{1.32}$$

where $p > 0$, $\tilde p > 0$, $\tau, \tilde\tau \in \mathfrak{M}(\Omega;\delta^\alpha)$ and $\nu, \tilde\nu \in \mathfrak{M}(\partial\Omega)$.

The next theorem provides a sufficient condition for the existence of solutions of (1.32).

Theorem E. *Let* $p > 0$, $\tilde p > 0$, $\tau, \tilde\tau \in \mathfrak{M}^+(\Omega;\delta^\alpha)$ *and* $\nu, \tilde\nu \in \mathfrak{M}^+(\partial\Omega)$. *Assume* $p\tilde p \neq 1$, $q < p_\mu$, $\mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu + \tilde\nu] \in L^{\tilde p}(\Omega,\delta^\alpha)$. *Then system* (1.32) *admits a weak solution* (u,v) *for* $\sigma > 0$ *and* $\tilde\sigma > 0$ *small if* $p\tilde p > 1$, *for any* $\sigma > 0$ *and* $\tilde\sigma > 0$ *if* $p\tilde p < 1$. *Moreover*

$$v \approx \mathbb{G}_\mu[\omega] + \mathbb{K}_\mu[\tilde\nu], \tag{1.33}$$

$$u \approx \mathbb{G}_\mu[(\mathbb{G}_\mu[\omega] + \mathbb{K}_\mu[\tilde\nu])^p] + \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu] \tag{1.34}$$

where the similarity constants depend on $N, p, \tilde p, \mu, \Omega, \sigma, \tilde\sigma, \tau, \tilde\tau$ *and*

$$\omega := \mathbb{G}_\mu[\tau + \mathbb{K}_\mu[\tilde\nu]^p]^{\tilde p} + \mathbb{K}_\mu[\nu]^{\tilde p} + \tilde\tau.$$

A new criterion for the existence of (1.32), expressed in terms of the capacities $\text{Cap}^{a}_{\mathbb{N}_{\theta,\beta},s}$ and $\text{Cap}^{\partial\Omega}_{\theta,s}$, is stated in the following result.

Theorem F. *Let $p > 1$, $\tilde{p} > 1$, $\tau, \tilde{\tau} \in \mathfrak{M}^+(\Omega; \delta^\alpha)$ and $\nu, \tilde{\nu} \in \mathfrak{M}^+(\partial\Omega)$. Assume there exists $C > 0$ such that*

$$\max\left\{\int_E \delta^\alpha d\tau, \int_E \delta^\alpha d\tilde{\tau}\right\} \le C \min\{\mathrm{Cap}^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2},p'}(E), \mathrm{Cap}^{(\tilde{p}+1)\alpha}_{\mathbb{N}_{2\alpha,2},\tilde{p}'}(E)\}, \ \forall E \subset \overline{\Omega}, \tag{1.35}$$

$$\max\{\nu(F), \tilde{\nu}(F)\} \le C \min\{\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{1+\alpha}{p},p'}(F), \mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{1+\alpha}{\tilde{p}},\tilde{p}'}(F)\}, \ \forall F \subset \partial\Omega. \tag{1.36}$$

Then (1.32) admits a weak solution (u, v) for $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$ small enough. There exists $C > 0$ such that

$$\begin{aligned} &\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] \le u \le C(\mathbb{G}_\mu[\sigma\tau + \tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\varrho\nu + \tilde{\varrho}\tilde{\nu}]),\\ &\mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}] \le v \le C(\mathbb{G}_\mu[\sigma\tau + \tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\varrho\nu + \tilde{\varrho}\tilde{\nu}]). \end{aligned} \tag{1.37}$$

Finally, we deal with elliptic systems with more general nonlinearities

$$\begin{cases} -L_\mu u = \epsilon\, g(v) + \sigma\tau & \text{in } \Omega,\\ -L_\mu v = \epsilon\, \tilde{g}(u) + \tilde{\sigma}\tilde{\tau} & \text{in } \Omega,\\ \mathrm{tr}\,(u) = \varrho\nu, \quad \mathrm{tr}\,(v) = \tilde{\varrho}\tilde{\nu} & \text{on } \partial\Omega \end{cases} \tag{1.38}$$

where g and $\tilde{g}$ are nondecreasing, continuous functions in $\mathbb{R}$, $\epsilon = \pm 1$, $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

We shall treat successively the cases $\epsilon = -1$ and $\epsilon = 1$. For any function f, define

$$\Lambda_f := \int_1^\infty s^{-1-p_\mu} |f(s) - f(-s)| ds \tag{1.39}$$

with p_μ defined in (1.8).

Theorem G. *Let $\epsilon = -1$ and $\sigma, \tilde{\sigma}, \varrho, \tilde{\varrho}$ be positive numbers, τ, $\tilde{\tau} \in \mathfrak{M}(\Omega; \delta^\alpha)$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$. Assume that $\Lambda_g + \Lambda_{\tilde{g}} < \infty$ and $g(s) = \tilde{g}(s) = 0$ for any $s \le 0$. Then system (1.38) admits a weak solution (u, v).*

When $\epsilon = 1$, different phenomenon occurs, which is reflected in the following result.

Theorem H. *Let $\epsilon = 1$, τ, $\tilde{\tau} \in \mathfrak{M}(\Omega; \delta^\alpha)$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$.*

I. SUBCRITICALITY. *Assume that $\Lambda_g + \Lambda_{\tilde{g}} < \infty$. In addition, assume that there exist $q_1 > 1$, $a_1 > 0$, $b_1 > 0$ such that*

$$|g(s)| \le a_1|s|^{q_1} + b_1 \quad \forall s \in [-1, 1], \tag{1.40}$$

$$|\tilde{g}(s)| \le a_1|s|^{q_1} + b_1 \quad \forall s \in [-1, 1]. \tag{1.41}$$

Then (1.38) admits a weak solution for $b_1, \sigma, \tilde{\sigma}, \varrho, \tilde{\varrho}$ small enough.

II. SUBLINEARITY. *Assume that there exist* $q_1 > 1$, $q_2 \in (0,1]$, $a_2 > 0$ *and* $b_2 > 0$ *such that* $\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|] \in L^{q_1}(\Omega; \delta^{\alpha-1})$ *and*

$$|g(s)| \leq a_2|s|^{q_1} + b_2 \quad \forall s \in \mathbb{R}, \tag{1.42}$$

$$|\tilde{g}(s)| \leq a_2|s|^{q_2} + b_2 \quad \forall s \in \mathbb{R}. \tag{1.43}$$

(a) *If* $q_1q_2 = 1$ *and* $a_2 > 0$ *is small then* (1.38) *admits a weak solution for any* $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

(b) *If* $q_1q_2 < 1$ *then* (1.38) *admits a weak solution for any* $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

III. SUBCRITICALITY AND SUBLINEARITY. *Assume that* $\Lambda_g < \infty$. *In addition, assume that there exist* $a_1 > 0$, $a_2 > 0$, $b_1 > 0$, $b_2 > 0$, $q_1 \in (1, p_\mu)$, $q_2 \in (0,1]$, *such that* (1.40) *and* (1.43) *hold.*

(a) *If* $q_1q_2 > 1$ *then* (1.38) *admits a weak solution for* $b_1, b_2, \sigma, \tilde{\sigma}, \varrho, \tilde{\varrho}$ *small enough.*

(b) *If* $q_2p_\mu = 1$ *and* a_2 *is mall enough then* (1.38) *admits a weak solution for any* $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

(c) *If* $q_2p_\mu < 1$ *then* (1.38) *admits a weak solution for every for any* $\sigma > 0$, $\tilde{\sigma} > 0$, $\varrho > 0$, $\tilde{\varrho} > 0$.

Remark about elliptic equations and systems with weights. We emphasize that Theorems B and C can be extended to the case of equations with weights of the form

$$-L_\mu u = \delta^\gamma u^p + \sigma\tau \quad \text{in } \Omega, \tag{1.44}$$

and Theorems E–H can be extended to the case of systems with weights of the form

$$\begin{cases} -L_\mu u = \epsilon\, \delta^\gamma g(v) + \sigma\tau & \text{in } \Omega, \\ -L_\mu v = \epsilon\, \delta^{\tilde{\gamma}} \tilde{g}(u) + \tilde{\sigma}\tilde{\tau} & \text{in } \Omega, \end{cases} \tag{1.45}$$

by using similar arguments. However, in order to avoid the complication of the proofs, we state and prove the results without weights.

The paper is organized as follows. In Section 2 we investigate properties of the boundary trace defined in Definition 1.2 and prove Propositions A and D. Theorems B and C are proved in Section 3 due to estimates on Green kernel, Martin kernel and the capacities $\mathrm{Cap}^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2},p'}$ and $\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}$. In Section 4 sufficient conditions for the existence of weak solutions to elliptic systems with power source terms (1.32) (Theorems E and F) are obtained by combining the method in [6] and the capacity approach. Finally, in Section 5, we establish existence results for elliptic systems with more general nonlinearities (Theorems G and H) due to Schauder fixed point theorem.

Notations. Throughout this paper, C, c, c', ... denotes positive constants which may vary from one appearance to another. The notation $A \approx B$ means $c^{-1}B \leq A \leq cB$ for some constant $c > 1$ depending on some structural constant.

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2. Preliminaries

2.1. Green kernel and Martin kernel

Denote by $L^p_w(\Omega;\tau)$, $1 \le p < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^p space (or Marcinkiewicz space) (see [17]). When $\tau = \delta^s dx$, for simplicity, we use the notation $L^p_w(\Omega;\delta^s)$. Notice that, for every $s > -1$,

$$L^p_w(\Omega;\delta^s) \subset L^r(\Omega;\delta^s) \quad \forall r \in [1,p). \tag{2.1}$$

Moreover for any $u \in L^p_w(\Omega;\delta^s)$ $(s > -1)$,

$$\int_{\{|u|\ge\lambda\}} \delta^s dx \le \lambda^{-p} \|u\|^p_{L^p_w(\Omega;\delta^s)} \quad \forall \lambda > 0. \tag{2.2}$$

Let G^Ω_μ and K^Ω_μ be respectively the Green kernel and Martin kernel of $-L_\mu$ in Ω (see [14,9]) for more details). We recall that

$$G^\Omega_\mu(x,y) \approx \min\left\{|x-y|^{2-N}, \delta(x)^\alpha \delta(y)^\alpha |x-y|^{2-N-2\alpha}\right\} \quad \forall x,y \in \Omega, x \ne y, \tag{2.3}$$

$$K^\Omega_\mu(x,y) \approx \delta(x)^\alpha |x-y|^{2-N-2\alpha} \quad \forall x \in \Omega,\ y \in \partial\Omega. \tag{2.4}$$

Finally, we denote by $\mathbb{G}_\mu$ and $\mathbb{K}_\mu$ be the corresponding Green operator and Martin operator (see [14,9]), namely

$$\mathbb{G}_\mu[\tau](x) = \int_\Omega G_\mu(x,y)d\tau(y), \quad \forall \tau \in \mathfrak{M}(\Omega), \tag{2.5}$$

$$\mathbb{K}_\mu[\nu](x) = \int_{\partial\Omega} K_\mu(x,z)d\nu(z), \quad \forall \nu \in \mathfrak{M}(\partial\Omega). \tag{2.6}$$

Let us recall a result from [4] which will be useful in the sequel.

Proposition 2.1. ([4, Lemma 2.4]) *Let ω be a nonnegative bounded Radon measure in $D = \Omega$ or $\partial\Omega$ and $\eta \in C(\Omega)$ be a positive weight function. Let H be a continuous nonnegative function on $\{(x,y) \in \Omega \times D : x \ne y\}$. For any $\lambda > 0$ we set*

$$A_\lambda(y) := \{x \in \Omega \setminus \{y\} : \ H(x,y) > \lambda\} \quad \textit{and} \quad m_\lambda(y) := \int_{A_\lambda(y)} \eta(x)dx.$$

Suppose that there exist $C > 0$ and $k > 1$ such that $m_\lambda(y) \le C\lambda^{-k}$ for every $\lambda > 0$. Then the operator

$$\mathbb{H}[\omega](x) := \int_D H(x, y) d\omega(y)$$

belongs to $L^k_w(\Omega; \eta)$ and

$$||\mathbb{H}[\omega]||_{L^k_w(\Omega;\eta)} \le (1 + \frac{Ck}{k-1})\omega(D).$$

By combining (2.3), (2.4) and the above Lemma we have the following result.

Lemma 2.2. *Let $\gamma \in \left(-\frac{\alpha N}{N+2\alpha-2}, \frac{\alpha N}{N-2}\right)$. Then there exists $C = C(N, \mu, \gamma, \Omega) > 0$ such that*

$$\sup_{\xi\in\Omega} \left|\left| \frac{G_\mu(\cdot, \xi)}{\delta(\xi)^\alpha} \right|\right|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega;\delta^\gamma)} < C. \tag{2.7}$$

Proof. Let $\xi \in \Omega$. We will apply Proposition 2.1 with $D = \Omega$, $\eta = \delta^\gamma$ with $\gamma > -1$, $\omega = \delta^\alpha \delta_\xi$, where δ_ξ is the Dirac measure concentrated at ξ, and

$$H(x, y) = \frac{G_\mu(x, y)}{\delta(y)^\alpha}.$$

Then

$$\mathbb{H}[\omega](x) = \int_\Omega \frac{G_\mu(x, y)}{\delta(y)^\alpha} \delta(y)^\alpha d\delta_\xi(y) = G_\mu(x, \xi).$$

From (2.3), there exists $C = C(N, \mu, \Omega)$ such that, for every $(x, y) \in \Omega \times \Omega$, $x \neq y$,

$$G_\mu(x, y) \le C\delta(y)^\alpha |x - y|^{2-N-\alpha}, \tag{2.8}$$

$$G_\mu(x, y) \le C\frac{\delta(y)^\alpha}{\delta(x)^\alpha} |x - y|^{2-N}, \tag{2.9}$$

$$G_\mu(x, y) \le C\delta(x)^\alpha \delta(y)^\alpha |x - y|^{2-N-2\alpha}. \tag{2.10}$$

By (2.8), for any $x \in A_\lambda(y)$,

$$\lambda \le C|x - y|^{2-N-\alpha}, \tag{2.11}$$

and form (2.9) and (2.10)

$$\delta(x)^\alpha \le \frac{C}{\lambda}|x - y|^{2-N} \quad \text{and} \quad \delta(x)^\alpha \ge C\lambda|x - y|^{N+2\alpha-2} \tag{2.12}$$

We consider two cases: $\gamma \geq 0$ and $-1 < \gamma < 0$.

Case 1: $\gamma \geq 0$. Due to (2.11) and (2.12) we have

$$m_\lambda(y) = \int\limits_{A_\lambda(y)} \delta(x)^\gamma dx \leq \int\limits_{A_\lambda(y)} \left(\frac{C}{\lambda}|x-y|^{2-N}\right)^{\frac{\gamma}{\alpha}} dx \leq C\lambda^{-\frac{N+\gamma}{N+\alpha-2}},$$

with $\gamma < \frac{\alpha N}{N-2}$. Observe that $\omega(\Omega) = \delta(\xi)^\alpha$, by Proposition 2.1, we get

$$\left|\left|G_\mu(\cdot,\xi)\right|\right|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega;\,\delta^\gamma)} \leq C\delta(\xi)^\alpha.$$

This implies (2.7).

Case 2: $-1 < \gamma < 0$. By (2.11) and (2.12) we have

$$m_\lambda(y) = \int\limits_{A_\lambda(y)} \delta(x)^\gamma dx \leq \int\limits_{A_\lambda(y)} (C\lambda|x-y|^{N+2\alpha-2})^{\frac{\gamma}{\alpha}} dx \leq C\lambda^{-\frac{N+\gamma}{N+\alpha-2}},$$

with $\gamma > -\frac{\alpha N}{N+2\alpha-2}$. By arguing similarly as in Case 1, we get (2.7). $\square$

Lemma 2.3. *Let $\gamma > -1$. Then there exists $C = C(N,\mu,\gamma,\Omega) > 0$ such that*

$$\sup_{\xi\in\partial\Omega} \left|\left|K_\mu(\cdot,\xi)\right|\right|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega;\delta^\gamma)} < C.$$

Proof. Let $\xi \in \partial\Omega$. We will apply Proposition 2.1 with $D = \partial\Omega$, $\eta = \delta^\gamma$ with $\gamma > -1$ and $\omega = \delta_\xi$. The rest of the proof can be proceeded as in the proof of Lemma 2.2 and we omit it. $\square$

In view of (2.1), Lemma 2.2 and Lemma 2.3, one can obtain easily the following proposition (see also [14,18]).

Proposition 2.4. (i) *Let $\gamma \in (-\frac{\alpha N}{N+2\alpha-2}, \frac{\alpha N}{N-2})$. Then there exists a constant $c = c(N,\mu,\gamma,\Omega)$ such that*

$$\left\|\mathbb{G}_\mu[\tau]\right\|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega;\delta^\gamma)} \leq c\,\|\tau\|_{\mathfrak{M}(\Omega;\delta^\alpha)} \quad \forall\tau \in \mathfrak{M}(\Omega;\,\delta^\alpha). \tag{2.13}$$

(ii) *Let $\gamma > -1$. Then there exists a constant $c = c(N,\mu,\gamma,\Omega)$ such that*

$$\left\|\mathbb{K}_\mu[\nu]\right\|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega;\delta^\gamma)} \leq c\,\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \forall\nu \in \mathfrak{M}(\partial\Omega). \tag{2.14}$$

2.2. Boundary trace

In this section we study properties of the boundary trace in connection with of L_μ harmonic functions. In particular, we show that, when $\mu < C_H(\Omega)$, the boundary trace defined in Definition 1.2 coincides the notion of normalized boundary trace introduced in Definition 1.1). To this end, we will examine that $\operatorname{tr}(\mathbb{G}_\mu[\tau]) = 0$ for every $\tau \in \mathfrak{M}(\Omega; \delta^\alpha)$ and $\operatorname{tr}(\mathbb{K}_\mu[\nu]) = \nu$ for every $\nu \in \mathfrak{M}(\partial\Omega)$. These results are proved below, based on a combination of the ideas in [9] and [14]. It is worth emphasizing that the below results are valid for $\mu \in (0, \frac{1}{4}]$ (under the condition that the first eigenvalue λ_μ of $-L_\mu$ is positive).

Proposition 2.5. *Let $\tau \in \mathfrak{M}(\Omega; \delta^\alpha)$ and $u = \mathbb{G}_\mu[\tau]$. Then $\operatorname{tr}(u) = 0$.*

Proof. First we assume that τ is nonnegative. Let $\{\Omega_n\}$ be a smooth exhaustion of Ω and for each n, let $\omega_{\Omega_n}^{x_0}$ be the $L_\mu^{\Omega_n}$ harmonic measure on $\partial\Omega_n$. Then u satisfies

$$\begin{cases} -L_\mu u = \tau & \text{in } \Omega_n \\ u = u & \text{on } \partial\Omega_n. \end{cases} \tag{2.15}$$

Thus

$$u = \mathbb{G}_\mu^{\Omega_n}[\tau] + \mathbb{K}_\mu^{\Omega_n}[u] = \mathbb{G}_\mu^{\Omega_n}[\tau] + \int_{\partial\Omega_n} u\, d\omega_{\Omega_n}^{x_0}. \tag{2.16}$$

This, joint with $\mathbb{G}_\mu^{\Omega_n}[\tau] \uparrow \mathbb{G}_\mu[\tau]$ as $n \to \infty$, ensures

$$\lim_{n\to\infty} \int_{\partial\Omega_n} u\, d\omega_{\Omega_n}^{x_0} = 0,$$

namely $\operatorname{tr}(u) = 0$.

In the general case, the result follows from the linearity property of the problem. $\square$

The next result shows that the boundary trace of L_μ harmonic function can be achieved in a *dynamic way*.

Proposition 2.6. *[9, Proposition 2.34] Let $x_0 \in \Omega_1$ and $\mu \in \mathfrak{M}(\partial\Omega)$. Put*

$$v(x) := \int_{\partial\Omega} K_\mu(x, y)\, d\nu(y),$$

then for every $\zeta \in C(\overline{\Omega})$,

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \zeta v\, d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega} \zeta\, d\nu. \tag{2.17}$$

Also we have the following representation formula for L_μ harmonic functions.

Proposition 2.7. *[9, Theorem 2.33] Let u be a positive L_μ harmonic in Ω. Then $u \in L^1(\Omega;\delta^\alpha)$ and there exists a unique Radon measure ν on $\partial\Omega$ such that*

$$u(x) = \int_{\partial\Omega} K_\mu(x,y)d\nu(y). \tag{2.18}$$

In the following proposition, we study the boundary trace of L_μ subharmonic functions.

Proposition 2.8. *Let w be a nonnegative L_μ subharmonic function. If w is dominated by an L_μ superharmonic function then $L_\mu w \in \mathfrak{M}^+(\Omega;\delta^\alpha)$ and w has a boundary trace $\nu \in \mathfrak{M}(\partial\Omega)$. In addition, if* $\operatorname{tr}(w) = 0$ *then* $w = 0$.

Proof. By proceeding as in the proof of [14, Proposition 2.14] and using Proposition 2.7, we obtain the desired result. □

Proposition 2.9. *Let w be a nonnegative L_μ subharmonic function. If w has a boundary trace then it is dominated by an L_μ harmonic function.*

Proof. The proof is similar to that of Proposition 2.20 in [14]. For the sake of convenience we give it below. Let $\{\Omega_n\}$ be as in the proof of Proposition 2.5 and fix $x_0 \in \Omega_1$. For any $x \in \Omega$, set

$$u_n(x) = \int_{\partial\Omega_n} w\, d\omega^x_{\Omega_n},$$

then u_n is $L_\mu^{\Omega_n}$ harmonic function with boundary trace w. Furthermore, by the maximum principle we have that $w \le u_n$ in Ω_n. Let $\nu \in \mathfrak{M}(\partial\Omega)$ be such that

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \zeta w\, d\omega^{x_0}_{\Omega_n} = \int_{\partial\Omega} \zeta\, d\nu \quad \forall \zeta \in C(\overline{\Omega}). \tag{2.19}$$

Then

$$u_n(x_0) = \int_{\partial\Omega_n} w\, d\omega^{x_0}_{\Omega_n} \to \int_{\partial\Omega} d\nu.$$

We infer from Harnack inequality that $\{u_n\}$ is locally uniformly bounded and hence there exists an L_μ harmonic function u such that $u_n \to u$ locally uniformly in Ω. By Proposition 2.8, there exists a nonnegative measure $\tau \in \mathfrak{M}^+(\Omega;\delta^\alpha)$ such that

$$w = -\mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu].$$

On the other hand,

$$w = -\mathbb{G}_\mu^{\Omega_n}[\tau] + u_n \to -\mathbb{G}_\mu[\tau] + u,$$

locally uniformly in Ω. Thus we can deduce that $u = \mathbb{K}_\mu[\nu]$ and the result follows. □

Proposition 2.10. *Let u be a nonnegative L_μ superharmonic function. Then there exist $\nu \in \mathfrak{M}^+(\partial\Omega)$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$ such that*

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu].$$

Proof. Let Ω_n and $\omega_{\Omega_n}^{x_0}$ be as in the proof of Proposition 2.5. Since u is L_μ superharmonic function there exists a nonnegative Radon measure in Ω such that

$$-L_\mu u = \tau \quad \text{in } \Omega$$

in the sense of distributions. Note that u is the unique solution of

$$\begin{cases} -L_\mu w = \tau & \text{in } \Omega_n \\ w = u & \text{on } \partial\Omega_n. \end{cases} \tag{2.20}$$

Therefore

$$u = \mathbb{G}_\mu^{\Omega_n}[\tau] + \mathbb{K}_\mu^{\Omega_n}[u]. \tag{2.21}$$

Set $w_n = \mathbb{K}_\mu^{\Omega_n}[u]$. Since $\tau \geq 0$, by the above quality, we have $0 \leq w_n(x) \leq u(x)$. Thus by the Harnack inequality, $w_n \to w$ locally uniformly in Ω. Furthermore, w is an L_μ harmonic function in Ω and by Proposition 2.18 there exists $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that

$$w = \mathbb{K}_\mu[\nu]. \tag{2.22}$$

Now since $G_\mu^{\Omega_n} \uparrow G_\mu$ as $n \to \infty$, we deduce from (2.21) and (2.22) that

$$u = \mathbb{G}_\mu^{\Omega_n}[\tau] + \mathbb{K}_\mu^{\Omega_n}[u] \to \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu].$$

Since

$$G_\mu(x, y) \geq c(x, \mu, N)\delta(y)^\alpha,$$

we can easily prove that $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$ which completes the proof. □

The above results enable to study the boundary value problem for the linear equation

$$\begin{cases} -L_\mu u = \tau & \text{in } \Omega, \\ \operatorname{tr}(u) = \nu. \end{cases} \tag{2.23}$$

Definition 2.11. Let $(\tau, \nu) \in \mathfrak{M}(\Omega; \delta^\alpha) \times \mathfrak{M}(\partial\Omega)$. We say that u is a *weak solution* of (2.23) if $u \in L^1(\Omega; \delta^\alpha)$ and

$$-\int_\Omega u L_\mu \zeta \, dx = \int_\Omega \zeta \, d\tau - \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta \, dx \qquad \forall \zeta \in \mathbf{X}_\mu(\Omega), \tag{2.24}$$

Proposition 2.12. *For any* $(\tau, \nu) \in \mathfrak{M}(\Omega; \delta^\alpha) \times \mathfrak{M}(\partial\Omega)$ *there exists a unique weak solution of* (2.23). *Moreover*

$$u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu], \tag{2.25}$$

$$\|u\|_{L^1(\Omega;\delta^\alpha)} \le c(\|\tau\|_{\mathfrak{M}(\Omega;\delta^\alpha)} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}). \tag{2.26}$$

In addition, for any $\zeta \in \mathbf{X}_\mu(\Omega)$, $\zeta \ge 0$,

$$-\int_\Omega |u| L_\mu \zeta \, dx \le \int_\Omega \zeta \operatorname{sign}(u) \, d\tau - \int_\Omega \mathbb{K}_\mu[|\nu|] L_\mu \zeta \, dx, \tag{2.27}$$

and

$$-\int_\Omega u_+ L_\mu \zeta \, dx \le \int_\Omega \zeta \operatorname{sign}_+(u) \, d\tau - \int_\Omega \mathbb{K}_\mu[\nu_+] L_\mu \zeta \, dx. \tag{2.28}$$

Proof. The proof is similar to that of [9, Proposition 3.2] and we omit it. □

Remark 2.1. If $h \in L^1(\partial\Omega, d\omega_\Omega^{x_0})$ is the boundary value of (2.23), the above Proposition is valid for $d\nu = h d\omega_\Omega^{x_0}$.

Proposition 2.13. *(i) For* $\tau \in \mathfrak{M}(\Omega; \delta^\alpha)$, $\operatorname{tr}(\mathbb{G}_\mu[\tau]) = 0$ *and for* $\nu \in \mathfrak{M}(\partial\Omega)$, $\operatorname{tr}(\mathbb{K}_\mu[\nu]) = \nu$.

(ii) Let w *be a nonnegative* L_μ *subharmonic function in* Ω. *Then* w *is dominated by an* L_μ *superharmonic function if and only if* w *has a boundary trace* $\nu \in \mathfrak{M}(\partial\Omega)$. *Moreover, if* w *has a boundary trace then* $L_\mu w \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. *If, in addition, if* $\operatorname{tr}(w) = 0$ *then* $w = 0$.

(iii) Let u *be a nonnegative* L_μ *superharmonic function. Then there exist* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *and* $\tau \in \mathfrak{M}^+(\Omega, \delta^\alpha)$ *such that* (2.25) *holds.*

(iv) Let $(\tau, \nu) \in \mathfrak{M}(\Omega; \delta^\alpha) \times \mathfrak{M}(\partial\Omega)$. *Then there exists a unique weak solution* u *of* (2.23). *The solution is given by* (2.25). *Moreover, there exists* $c = c(N, \mu, \Omega)$ *such that* (2.26) *holds.*

Proof. Statement (i) follows from Proposition 2.5 and Proposition 2.6. Statement (ii) can be deduced from Proposition 2.8 and Proposition 2.9. Statement (iii) follows from Proposition 2.10. Finally statement (iv) is obtained due to Proposition 2.12. □

Proof of Proposition A. We infer from [9] that $(i) \Longleftrightarrow (ii)$. By an argument similar to that of the proof of [18, Theorem B], we deduce that $(ii) \Longleftrightarrow (iii)$. □

For $\beta > 0$, put

$$\Omega_\beta := \{x \in \Omega : \delta(x) < \beta\}, \; D_\beta := \{x \in \Omega : \delta(x) > \beta\}, \; \Sigma_\beta := \{x \in \Omega : \delta(x) = \beta\}. \tag{2.29}$$

Lemma 2.14. *There exists $\beta_* > 0$ such that for every point $x \in \overline{\Omega}_{\beta_*}$, there exists a unique point $\sigma_x \in \partial\Omega$ such that $x = \sigma_x - \delta(x)\mathbf{n}_{\sigma_x}$. The mappings $x \mapsto \delta(x)$ and $x \mapsto \sigma_x$ belong to $C^2(\overline{\Omega}_{\beta_*})$ and $C^1(\overline{\Omega}_{\beta_*})$ respectively. Moreover,* $\lim_{x\to\sigma(x)} \nabla\delta(x) = -\mathbf{n}_{\sigma_x}$.

Proof of Proposition D. (iii) $\Longrightarrow$ (ii). Assume (u, v) is a distribution solution of (1.30). Put $\omega := g(v)$ and denote $\omega_\beta := \omega|_{D_\beta}$, $\tau_\beta := \tau|_{D_\beta}$ and $\lambda_\beta := u|_{\Sigma_\beta}$ for $\beta \in (0, \beta_*)$. Consider the boundary value problem

$$-L_\mu w = \omega_\beta + \tau_\beta \quad \text{in } D_\beta, \qquad w = \lambda_\beta \quad \text{on } \Sigma_\beta.$$

This problem admits a unique solution w_β (see [9]). Therefore $w_\beta = u|_{D_\beta}$. We have

$$u|_{D_\beta} = w_\beta = \mathbb{G}_\mu^{D_\beta}[\omega_\beta] + \mathbb{G}_\mu^{D_\beta}[\tau_\beta] + \mathbb{P}_\mu^{D_\beta}[\lambda_\beta]$$

where $\mathbb{G}_\mu^{D_\beta}$ and $\mathbb{P}_\mu^{D_\mu}$ are respectively Green kernel and Poisson kernel of $-L_\mu$ in D_β.

It follows that

$$\left| \int_{D_\beta} G_\mu^{D_\beta}(\cdot, y) g(v(y))\, dy \right| = \left| \mathbb{G}_\mu^{D_\beta}[\tau_\beta] \right| \le \left| u|_{D_\beta} \right| + \left| \mathbb{G}_\mu^{\Omega}[\tau] \right| + \left| \mathbb{K}_\mu^{\Omega}[\nu] \right|.$$

Letting $\beta \to 0$, we get

$$\left| \int_{\Omega} G_\mu(\cdot, y) g(v(y))\, dy \right| < \infty. \tag{2.30}$$

Fix a point $x_0 \in \Omega$. Keeping in mind that $G_\mu(x_0, y) \approx \delta(y)^\alpha$ for every $y \in \Omega_{\beta_*}$, we deduce from (2.30) that $g(v) \in L^1(\Omega; \delta^\alpha)$. Similarly, one can show that $\tilde{g}(u) \in L^1(\Omega; \delta^\alpha)$. Thanks to Proposition 2.13 (v), we obtain (1.29).

(ii) $\Longrightarrow$ (iii). Assume u and v are functions such that $\tilde{g}(u) \in L^1(\Omega; \delta^\alpha)$, $g(v) \in L^1(\Omega; \delta^\alpha)$ and (1.29) holds. By Proposition 2.13 (i) $L_\mu \mathbb{K}_\mu[\nu] = L_\mu \mathbb{K}_\mu[\tilde{\nu}] = 0$, which implies that (u, v) is a solution of (1.30). On the other hand, since $\tilde{g}(u) \in L^1(\Omega; \delta^\alpha)$ and $g(v) \in L^1(\Omega; \delta^\alpha)$, we deduce from Proposition 2.13 (ii) that $\mathrm{tr}\,(\mathbb{G}_\mu[\tilde{g}(u)]) = \mathrm{tr}\,(\mathbb{G}_\mu[g(v)]) = 0$. Consequently, $\mathrm{tr}\,(u) = \mathrm{tr}\,(\mathbb{K}_\mu[\nu]) = \nu$ and $\mathrm{tr}\,(v) = \mathrm{tr}\,(\mathbb{K}_\mu[\tilde{\nu}]) = \tilde{\nu}$.

(iii) $\Longrightarrow$ (i). Assume (u, v) is a positive solution of (1.30) in the sense of distributions. From the implication (iii) $\Longrightarrow$ (ii), we deduce that $u \in L^1(\Omega; \delta^\alpha)$, $v \in L^1(\Omega; \delta^\alpha)$, $\tilde{g}(u) \in L^1(\Omega; \delta^\alpha)$ and $g(v) \in L^1(\Omega; \delta^\alpha)$. Hence, by Proposition 2.13, (1.28) holds for every $\phi \in \mathbf{X}_\mu(\Omega)$.

(i) $\Longrightarrow$ (iii). This implication follows straightforward from Proposition 2.13. □

3. The scalar problem

3.1. Concavity properties and Green properties

Here we give some concavity lemmas that will be employed in the sequel.

Proposition 3.1. *Let $\varphi \in L^1(\Omega;\delta^\alpha)$, $\varphi \geq 0$ and $\tau \in \mathfrak{M}^+(\Omega;\delta^\alpha)$. Set*

$$w := \mathbb{G}_\mu[\varphi+\tau] \quad \textit{and} \quad \psi = \mathbb{G}_\mu[\tau].$$

Let ϕ be a concave nondecreasing C^2 function on $[0,\infty)$, such that $\phi(1)\geq 0$. Then $\phi'(w/\psi)\varphi \in L^1(\Omega;\delta^\alpha)$ and the following holds in the weak sense in Ω

$$-L_\mu(\psi\phi(w/\psi)) \geq \phi'(w/\psi)\varphi.$$

Proof. Let $\{\varphi_n\}, \tau_n \in C^\infty(\overline{\Omega})$ such that $\varphi_n \to \varphi$ in $L^1(\Omega,\delta^\alpha)$ and $\tau_n \rightharpoonup \tau$. Set $w_n := \mathbb{G}_\mu[\varphi_n + \tau_n]$ and $\psi_n = \mathbb{G}_\mu[\tau_n]$. Since $w_n \geq \psi_n > 0$ for any $n \geq n_0$ for some $n_0 \in \mathbb{N}$, we have by straightforward calculations

$$\begin{aligned}-\Delta\left(\psi_n\phi(\frac{w_n}{\psi_n})\right) &= (-\Delta\psi_n)\left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n}\phi'(\frac{w_n}{\psi_n})\right) + (-\Delta w_n)\phi'(\frac{w_n}{\psi_n})\\ &\quad - \psi_n\phi''(\frac{w_n}{\psi_n})\left|\nabla\left(\frac{w_n}{\psi_n}\right)\right|^2.\end{aligned}$$

Now note that, since $\phi' \geq 0$, we have

$$(-\Delta w_n)\phi'(\frac{w_n}{\psi_n}) \geq \phi'(\frac{w_n}{\psi_n})\left(-\Delta\psi_n - \mu\frac{\psi_n}{\delta^2} + \mu\frac{w_n}{\delta^2} + \varphi_n\right).$$

This, together with the fact that $\phi(t) - t\phi'(t) + \phi'(t) \geq 0$ for any $t \geq 1$, implies

$$\begin{aligned}&(-\Delta\psi_n)\left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n}\phi'(\frac{w_n}{\psi_n})\right) + (-\Delta w_n)\phi'(\frac{w_n}{\psi_n})\\ &\geq (-\Delta\psi_n)\left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n}\phi'(\frac{w_n}{\psi_n}) + \phi'(\frac{w_n}{\psi_n})\right) + \phi'(\frac{w_n}{\psi_n})\left(-\mu\frac{\psi_n}{\delta^2} + \mu\frac{w_n}{\delta^2} + \varphi_n\right)\\ &\geq \mu\frac{\psi_n}{\delta^2}\left(\phi(\frac{w_n}{\psi_n}) - \frac{w_n}{\psi_n}\phi'(\frac{w_n}{\psi_n}) + \phi'(\frac{w_n}{\psi_n})\right) + \phi'(\frac{w_n}{\psi_n})\left(-\mu\frac{\psi_n}{\delta^2} + \mu\frac{w_n}{\delta^2} + \varphi_n\right)\\ &= \frac{\mu}{\delta^2}\psi_n\phi(\frac{w_n}{\psi_n}) + \phi'(\frac{w_n}{\psi_n})\varphi_n.\end{aligned}$$

Thus we have proved

$$-L_\mu\left(\psi_n\phi(\frac{w_n}{\psi_n})\right) \geq \phi'(\frac{w_n}{\psi_n})\varphi_n.$$

Also

$$\psi_n\phi(\frac{w_n}{\psi_n}) \leq \psi_n(\phi(0) + \phi'(0)\frac{w_n}{\psi_n}) \leq C(\psi_n + w_n)$$

and

$$-\int_\Omega \psi_n\phi(\frac{w_n}{\psi_n})L_\mu\xi\, dx \geq \int_\Omega \phi'(\frac{w_n}{\psi_n})\varphi_n\xi\, dx \quad \forall \xi \in \mathbf{X}_\mu(\Omega).$$

By passing to the limit with Lebesgue theorem and Fatou lemma, we complete the proof. □

In the next Lemma we will prove the 3-G inequality which will be useful later.

Lemma 3.2. *There exists a positive constant $C = C(N, \mu, \Omega)$ such that*

$$\frac{G_\mu(x,y)G_\mu(y,z)}{G_\mu(x,z)} \leq C\left(\frac{\delta(y)^\alpha}{\delta(x)^\alpha}G_\mu(x,y) + \frac{\delta(y)^\alpha}{\delta(z)^\alpha}G_\mu(y,z)\right) \quad \forall (x,y,z) \in \Omega\times\Omega\times\Omega. \tag{3.1}$$

Proof. It follows from (2.3) and the inequality $|\delta(x) - \delta(y)| \leq |x-y|$ that

$$\begin{aligned} G_\mu(x,y) &\approx \min\left\{|x-y|^{2-N}, \delta(x)^\alpha\delta(y)^\alpha\,|x-y|^{2-2\alpha-N}\right\} \\ &\approx |x-y|^{2-N}\,\delta(x)^\alpha\delta(y)^\alpha\left(\max\left\{\delta(x)^\alpha\delta(y)^\alpha, |x-y|^{2\alpha}\right\}\right)^{-1} \\ &\approx |x-y|^{2-N}\,\delta(x)^\alpha\delta(y)^\alpha\,(\max\{\delta(x),\delta(y),|x-y|\})^{-2\alpha} \\ &= \delta(x)^\alpha\delta(y)^\alpha N_{2\alpha,2}(x,y), \quad \forall x,y \in \Omega,\ x \neq y, \end{aligned}$$

where $N_{2\alpha,2}(x,y)$ is defined in (1.23) with $a = 2\alpha$ and $\beta = 2$. By [5, Lemma 2.2] we deduce that there exists a positive constant $C = C(N,\mu,\Omega)$ such that

$$\frac{1}{N_{2\alpha,2}(x,z)} \leq C\left(\frac{1}{N_{2\alpha,2}(x,y)} + \frac{1}{N_{2\alpha,2}(y,z)}\right). \tag{3.2}$$

From (3.2) we can easily obtain (3.1). □

Lemma 3.3. *Let $0 < p < p_\mu$ and $\tau \in \mathfrak{M}^+(\Omega;\delta^\alpha)$. Then there is a constant $C = C(N,\mu,p,\tau,\Omega) > 0$ such that (1.19) holds.*

Proof. First we assume that $p > 1$. By (2.13) we have that $\mathbb{G}_\mu[\tau]^p \in L^1(\Omega;\delta^\alpha)$. We write

$$\mathbb{G}_\mu[\tau](y) = \int_\Omega G_\mu(y,z)d\tau(z) = \int_\Omega \frac{G_\mu(y,z)}{\delta(z)^\alpha}\delta(z)^\alpha d\tau(z),$$

thus

$$\mathbb{G}_\mu[\tau](y)^p \leq C\int_\Omega \delta(z)^\alpha\left(\frac{G_\mu(y,z)}{\delta(z)^\alpha}\right)^p d\tau(z).$$

Consequently,

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p](x) \le C \int_\Omega \int_\Omega G_\mu(x,y) G_\mu(y,z)^p \delta(z)^{\alpha(1-p)} d\tau(z) dy. \tag{3.3}$$

Also by (3.1) we obtain

$$\begin{aligned}
&\int_\Omega \int_\Omega G_\mu(x,y) G_\mu(y,z)^p \delta(z)^{\alpha(1-p)} d\tau(z) dy \\
&\le C \int_\Omega G_\mu(x,z) \int_\Omega \delta(y)^\alpha \left(\left(\frac{G_\mu(x,y)}{\delta(x)^\alpha} \right) \left(\frac{G_\mu(y,z)}{\delta(z)^\alpha} \right)^{p-1} + \left(\frac{G_\mu(y,z)}{\delta(z)^\alpha} \right)^p \right) dy d\tau(z) \\
&\le C \int_\Omega G_\mu(x,z) \int_\Omega \delta(y)^\alpha \left(\left(\frac{G_\mu(x,y)}{\delta(x)^\alpha} \right)^p + \left(\frac{G_\mu(y,z)}{\delta(z)^\alpha} \right)^p \right) dy d\tau(z),
\end{aligned} \tag{3.4}$$

where in the last inequality we have used the Hölder inequality. By (3.3), (3.4) and Lemma 2.2 we derive that

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p](x) \le C \int_\Omega G_\mu(x,z) d\tau(z).$$

Note that the above argument is still valid for $p = 1$.

If $0 \le p < 1$ then

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] \le C(\mathbb{G}_\mu[1] + \mathbb{G}_\mu[\mathbb{G}_\mu[\tau]]).$$

By combining the case $p = 1$ and the estimate $\mathbb{G}_\mu[1] \le C\mathbb{G}_\mu[\tau]$, we obtain (1.19). □

Actually (1.19) is a sufficient condition for the existence of weak solution of

$$\begin{cases} -L_\mu u = u^p + \sigma\tau & \text{in } \Omega, \\ \operatorname{tr}(u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Proposition 3.4. *Let $0 < p \ne 1$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. Assume that there exists a positive constant C such that (1.19) holds. Then problem (3.5) admits a weak solution u satisfying*

$$\mathbb{G}_\mu[\sigma\tau] \le u \le C\mathbb{G}_\mu[\sigma\tau] \quad \text{a.e. in } \Omega, \tag{3.6}$$

with another constant $C > 0$, for any $\sigma > 0$ small enough if $p > 1$, for any $\sigma > 0$ if $p < 1$.

Proof. We adapt the idea in the proof of [6, Theorem 3.4]. Put $w := A\mathbb{G}_\mu[\sigma\tau]$ where $A > 0$ will be determined later. By (1.19),

$$\mathbb{G}_\mu[w^p + \sigma\tau] \le (CA^p\sigma^{p-1} + 1)\mathbb{G}_\mu[\sigma\tau] \quad \text{in } \Omega.$$

Therefore we deduce that $w \geq \mathbb{G}_\mu[w^p + \sigma\tau]$ as long as

$$CA^p\sigma^{p-1} + 1 \leq A. \tag{3.7}$$

If $p > 1$ then (3.7) holds if we choose $A > 1$ and then choose $\sigma > 0$ small enough. If $p \in (0, 1)$ then (3.7) holds if we choose $\sigma > 0$ arbitrary and then choose $A > 0$ large enough.

Next put $u_0 := \mathbb{G}_\mu[\sigma\tau]$ and $u_{n+1} := \mathbb{G}_\mu[u_n^p + \sigma\tau]$. It is clear that $\{u_n\}$ is increasing and $u_n \leq w$ in Ω for all n. Since (1.19) holds, $w^p \in L^1(\Omega; \delta^\alpha)$. Consequently, by monotone convergence theorem, there exists a function $u \in L^p(\Omega; \delta^\alpha)$ such that $u_n^p \to u^p$ in $L^1(\Omega; \delta^\alpha)$. It is easy to see that u is a solution of (3.5) satisfying (3.6). $\square$

Estimate (1.19) is also a necessary condition for the existence of weak solution of (3.5).

Proposition 3.5. *Let $p > 1$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. Assume that problem (3.5) admits a weak solution. Then (1.19) holds with $C = \frac{1}{p-1}$.*

Proof. We adapt the argument used in the proof of [6, Proposition 3.5]. Assume (3.5) has a solution $u \in L^p(\Omega; \delta^\alpha)$ and assume $\sigma = 1$. By applying Proposition 3.1 with φ replaced by u^p and with

$$\phi(s) = \begin{cases} (1 - s^{1-p})/(p-1) & \text{if } s \geq 1, \\ s - 1 & \text{if } s < 1, \end{cases}$$

we get (1.19) with $C = \frac{1}{p-1}$. $\square$

Proposition 3.6. *Let $0 < p < p_\mu$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. Then there exists a positive constant $C = C(N, \mu, \Omega, \sigma, \tau)$ such that for any weak solution u of (3.5) there holds*

$$\mathbb{G}_\mu[\sigma\tau] \leq u \leq C(\mathbb{G}_\mu[\sigma\tau] + \delta^\alpha) \quad \text{a.e. in } \Omega. \tag{3.8}$$

Proof. We follow the idea in the proof of [6, Theorem 3.6]. We may assume that $\sigma = 1$. If $0 \leq p < 1$, then

$$u = \mathbb{G}_\mu[u^p + \tau] \leq C(\mathbb{G}_\mu[1] + \mathbb{G}_\mu[u] + \mathbb{G}_\mu[\tau]).$$

Since $\mathbb{G}_\mu[1] \leq C\delta^\alpha$ a.e. in Ω, we obtain

$$u \leq C(\delta^\alpha + \mathbb{G}_\mu[u] + \mathbb{G}_\mu[\tau]) \quad \text{a.e. in } \Omega.$$

Therefore it is sufficient to deal with the term $\mathbb{G}_\mu[u]$ and we may assume that $p \geq 1$. Set

$$u_1 := u - \mathbb{G}_\mu[\tau] = \mathbb{G}_\mu[u^p],$$

hence $u = u_1 + \mathbb{G}_\mu[\tau]$. Since $u \in L^p(\Omega; \delta^\alpha)$ (by assumption), it follows that $u + \tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$, therefore by (2.13), $u \in L^s(\Omega; \delta^\alpha)$, for all $1 \leq s < p_\mu$. Thus there exists $k_0 > 1$ such that $u^p \in L^{k_0}(\Omega; \delta^\alpha)$.

Let $p < s < p_\mu$. By Hölder inequality we obtain

$$\begin{aligned} u_1(x)^{k_0 s} = \mathbb{G}_\mu[u^p](x)^{k_0 s} &= \left(\int_\Omega \frac{G_\mu(x,y)}{\delta(y)^\alpha} \delta(y)^\alpha u(y)^p dy \right)^{k_0 s} \\ &\le \left(\int_\Omega \frac{G_\mu(x,y)}{\delta(y)^\alpha} \delta(y)^\alpha u(y)^{k_0 p} dy \right)^s \left(\int_\Omega \frac{G_\mu(x,y)}{\delta(y)^\alpha} \delta(y)^\alpha dy \right)^{\frac{(k_0-1)s}{k_0}} \\ &\le C \int_\Omega \left(\frac{G_\mu(x,y)}{\delta(y)^\alpha} \right)^s \delta(y)^\alpha u(y)^{k_0 p} dy. \end{aligned}$$

This, joint with Lemma 2.2, yields

$$\int_\Omega u_1(y)^{k_0 s} \delta(y)^\alpha dy \le C \int_\Omega u(y)^{k_0 p} \delta(y)^\alpha \int_\Omega \left(\frac{G_\mu(x,y)}{\delta(y)^\alpha} \right)^s \delta(x)^\alpha dx dy < c.$$

Since $u^p \le C(u_1^p + \mathbb{G}_\mu[\tau]^p)$, by Lemma 3.3 we have

$$u \le C\left(\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] + u_2\right) + \mathbb{G}_\mu[\tau] \le C(\mathbb{G}_\mu[\tau] + u_2),$$

where $u_2 := \mathbb{G}_\mu[u_1^p]$. Note that $u_2 \in L^{\frac{k_0 s^2}{p^2}}(\Omega; \delta^\alpha)$.

By induction we define $u_n := \mathbb{G}_\mu[u_{n-1}^p]$ and we have $u \le C(\mathbb{G}_\mu[\tau] + u_n)$, $u_n^p \in L^{s_n}(\Omega; \delta^\alpha)$ with $s_n = \frac{k_0 s^n}{p^n}$. Since $s_n \to \infty$, by [17, Lemma 2.3.2] we have for $1 < s < p_\mu$,

$$\begin{aligned} u_n &\le C \int_\Omega |x-y|^{2-\alpha-N} u_{n-1}^p \delta(y)^\alpha dy \\ &\le C \left(\int_\Omega |x-y|^{(2-\alpha-N)s} \delta(y)^\alpha dy + \int_\Omega |x-y|^{2-\alpha-N} u_{n-1}^{\frac{ps}{s-1}} \delta(y)^\alpha dy \right) \\ &\le C', \end{aligned}$$

for n large enough. Therefore we obtain $u \le C(\mathbb{G}_\mu[\tau] + 1)$, which implies $u \le C(\mathbb{G}_\mu[\tau] + \mathbb{G}_\mu[1])$ with another $C > 0$. This, together with the inequality $\mathbb{G}_\mu[1] \le C\delta^\alpha$, implies (3.8). $\square$

3.2. New Green properties

Lemma 3.7. *Let $0 < p < p_\mu$, $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. Let s be such that*

$$\max\left(0, p - p_\mu + 1\right) < s \le 1. \tag{3.9}$$

Then there exists a constant $C > 0$ such that

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] \le C\mathbb{G}_\mu[\tau]^s \quad a.e. \text{ in } \Omega. \tag{3.10}$$

Proof. First we assume that $p > 1$. In view of the proof of Lemma 3.3, we have

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p](x) \le C \int_\Omega \int_\Omega G_\mu(x,y)G_\mu(y,z)^p \delta(z)^{\alpha(1-p)} d\tau(z)dy$$

$$= C \int_\Omega \int_\Omega G_\mu(x,y)^{1-s} G_\mu(x,y)^s G_\mu(y,z)^s \left(\frac{G_\mu(y,z)}{\delta^\alpha(z)}\right)^{p-s} \delta(z)^{\alpha(1-s)} d\tau(z)dy$$

$$\le C \int_\Omega G_\mu(x,z)^s \delta(z)^{\alpha(1-s)} \int_\Omega \delta(y)^\alpha \frac{G_\mu(x,y)}{\delta(x)} \left(\frac{G_\mu(y,z)}{\delta(z)^\alpha}\right)^{p-s} dyd\tau(z) \tag{3.11}$$

$$+ C \int_\Omega G_\mu(x,z)^s \delta(z)^{\alpha(1-s)} \int_\Omega \delta(y)^\alpha \left(\frac{G_\mu(x,y)}{\delta(x)}\right)^{1-s} \left(\frac{G_\mu(y,z)}{\delta(z)^\alpha}\right)^{p} dyd\tau(z) \tag{3.12}$$

$$\le C \int_\Omega G_\mu(x,z)^s \delta(z)^{\alpha(1-s)} \int_\Omega \delta(y)^\alpha \left(\frac{G_\mu(x,y)}{\delta(x)}\right)^{p-s+1} dyd\tau(z) \tag{3.13}$$

$$+ \int_\Omega G_\mu(x,z)^s \delta(z)^{\alpha(1-s)} \int_\Omega \delta(y)^\alpha \left(\frac{G_\mu(y,z)}{\delta(z)^\alpha}\right)^{p-s+1} dyd\tau(z) \tag{3.14}$$

$$\le C \int_\Omega \left(\frac{G_\mu(x,z)}{\delta(z)^\alpha}\right)^s \delta(z)^\alpha d\tau(z) \tag{3.15}$$

$$\le C \left(\int_\Omega G_\mu(x,z)d\tau(z)\right)^s. \tag{3.16}$$

Here (3.11) and (3.12) follow from (3.1), (3.13) and (3.14) follow from Hölder inequality, and (3.15) follows from Lemma 2.2, Hölder inequality and (3.9).

Note that the above approach can be applied to the case $p = 1$.

If $0 \le p < 1$ then

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] \le C(\mathbb{G}_\mu[1] + \mathbb{G}_\mu[\mathbb{G}_\mu[\tau]]) \le C(\mathbb{G}_\mu[1] + \mathbb{G}_\mu[\tau]^s)$$

Then (3.10) follows by a similar argument as in the proof of Lemma 3.3. □

3.3. Capacities and existence results

For $a \ge 0$, $0 \le \theta \le \beta < N$ and $s > 1$, let $N_{\theta,\beta}$, $\mathbb{N}_{\theta,\beta}$ and $\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}$ be defined as in (1.22), (1.23) and (1.24) respectively.

In this section we recall some results in [5, Section 2].

We recall below the definition of the capacity associated to $\mathbb{N}_{\theta,\beta}$ (see [11]).

Definition 3.8. Let $a \geq 0$, $0 \leq \theta \leq \beta < N$ and $s > 1$. Define $\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}$ by

$$Cap^a_{\mathbb{N}_{\theta,\beta},s}(E) := \inf\left\{ \int_{\overline{\Omega}} \delta^a \phi^s dy : \ \phi \geq 0, \ \ \mathbb{N}_{\theta,\beta}[\delta^a \phi] \geq \chi_E \right\},$$

for any Borel set $E \subset \overline{\Omega}$.

Clearly we have

$$\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E) = \inf\left\{ \int_{\overline{\Omega}} \delta^{-a(s-1)} \phi^s dy : \ \phi \geq 0, \ \ \mathbb{N}_{\theta,\beta}[\phi] \geq \chi_E \right\},$$

for any Borel set $E \subset \overline{\Omega}$. Furthermore we have by [1, Theorem 2.5.1]

$$\left(\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E)\right)^{\frac{1}{s}} = \inf\left\{ \omega(E) : \ \omega \in \mathfrak{M}^+_b(\overline{\Omega}), \ \left|\left|\mathbb{N}_{\theta,\beta}[\omega]\right|\right|_{L^{s'}(\overline{\Omega};\delta^a)} \leq 1 \right\}, \tag{3.17}$$

for any compact set $E \subset \overline{\Omega}$ where s' is the conjugate exponent of s.

Using [5, Theorem 2.6], we obtain easily the following result.

Proposition 3.9. *Let $p > 1$, $\sigma > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. Then the following statements are equivalent.*

1. There exists $C > 0$ such that the following inequality hold

$$\int_E \delta^\alpha d\tau \leq C \mathrm{Cap}^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2},p'}(E),$$

for any Borel $E \subset \overline{\Omega}$.

2. There exists a constant $C > 0$ such that (1.19) holds.

3. Problem (3.5) has a positive weak solution for $\sigma > 0$ small enough.

Proof. First we note that

$$G(x, y) = \delta(x)^\alpha \delta(y)^\alpha N_{2\alpha,2}(x, y), \quad \forall x, y \in \Omega, \ x \neq y.$$

Thus the inequality

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] \leq C \mathbb{G}_\mu[\tau] \quad \text{a.e. in } \Omega$$

is equivalent to

$$\mathbb{N}_{2\alpha,2}[\delta^{(p+1)\alpha}(y)\mathbb{N}_{2\alpha,2}[\tilde{\tau}]^p(y)](x) \leq C \mathbb{N}_{2\alpha,2}[\tilde{\tau}](x) \quad \text{a.e. in } \Omega,$$

where $d\tilde{\tau}(y) = \delta^\alpha(y) d\tau(y)$.

Now notice that if u is a positive solution of (3.5) then by Proposition 2.12 we have that $u = \mathbb{G}_\mu[u^p] + \mathbb{G}_\mu[\tau]$ which implies that

$$\frac{u}{\delta(x)^\alpha} \approx \mathbb{N}_{2\alpha,2}\left[\delta^{(p+1)\alpha}\left(\frac{u}{\delta^\alpha}\right)^p\right](x) + \sigma\mathbb{N}_{2\alpha,2}[\tilde{\tau}](x),$$

the desired results follow by [5, Theorem 2.6] and [5, Proposition 2.7]. □

Let us now give a result which implies the existence for the problem (3.5).

Lemma 3.10. *Let* $1 < p < p_\mu$. *Then*

$$\inf_{\xi\in\Omega} \mathrm{Cap}^{(p+1)\alpha}_{\mathbb{N}_{2\alpha,2,p'}}(\{\xi\}) > 0.$$

Proof. By (3.17) it is enough to show that

$$\sup_{\xi\in\Omega} \left|\left|\mathbb{N}_{2\alpha,2}[\delta_\xi]\right|\right|_{L^p(\Omega;\delta^{(p+1)\alpha})} < C < \infty,$$

which is equivalent to

$$\sup_{\xi\in\Omega} \left|\left|\frac{G_\mu(\cdot,\xi)}{\delta(\xi)^\alpha}\right|\right|_{L^p(\Omega;\delta^\alpha)} < C. \tag{3.18}$$

The result follows by Lemma 2.2 and (2.1). □

3.4. Boundary value problem

Estimate (1.18) is a necessary and sufficient condition for the existence of weak solutions of

$$\begin{cases} -L_\mu u = u^p & \text{in } \Omega, \\ \mathrm{tr}\,(u) = \varrho\nu & \text{on } \partial\Omega. \end{cases} \tag{3.19}$$

Proposition 3.11. *[5, Theorem 4.1] Let* $p > 1$, $\varrho > 0$ *and* $\nu \in \mathfrak{M}^+(\partial\Omega)$. *Then, the following statements are equivalent.*

1. There exists $C > 0$ *such that the following inequality holds*

$$\nu(F) \leq C\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{1+\alpha}{p},p'}(F)$$

for any Borel $F \subset \partial\Omega$.

2. There exists $C > 0$ *such that* (1.18) *holds.*

3. Problem (3.19) *has a positive weak solution for* $\varrho > 0$ *small enough.*

Lemma 3.12. *Let* $\nu \in \mathfrak{M}^+(\partial\Omega)$ *and* $0 < p < p_\mu$. *Then there exists a constant* $C > 0$ *such that* (1.18) *holds.*

Proof. We first assume that $1 < p < p_\mu$. Let $\xi \in \partial\Omega$; we have $\delta_\xi(F) < c\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{1+\alpha}{p},p'}(F)$ for every $F \subset \partial\Omega$ where c is independent of ξ. By Proposition 3.11, (1.18) holds with ν replaced by δ_ξ and with the constant C independent of ξ. By taking integral over $\xi \in \partial\Omega$, we get (1.18).

Next, if $p \in (0, 1]$, we choose $s > 1$ such that $1 < ps < p_\mu$. By Young's inequality,

$$\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^p] \le C(\mathbb{G}_\mu[1] + \mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^{ps}]) \le C(\mathbb{G}_\mu[1] + \mathbb{K}_\mu[\nu]). \tag{3.20}$$

This, combined with the inequality $\mathbb{G}_\mu[1] \le c\delta^\alpha \le c'\mathbb{K}_\mu[\nu]$ a.e. in Ω leads to (1.18). □

Proposition 3.13. *Let $p > 0$, $\varrho > 0$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$.*

(i) *Assume there exists a constant $C > 0$ such that* (1.18) *holds. Then problem* (3.19) *admits a weak solution u satisfying*

$$\mathbb{K}_\mu[\varrho\nu] \le u \le C\mathbb{K}_\mu[\varrho\nu] \quad \textit{a.e. in } \Omega, \tag{3.21}$$

with another constant $C > 0$, for any $\varrho > 0$ small enough if $p > 1$, for any $\varrho > 0$ if $p \in (0, 1)$.

(ii) *Assume $p > 1$ and problem* (3.19) *admits a weak solution. Then* (1.18) *holds with $C = \frac{1}{p-1}$.*

(iii) *Assume $0 < p < p_\mu$. Then there exists a constant $C > 0$ such that for any weak solution u of* (3.19) *there holds*

$$\mathbb{K}_\mu[\varrho\nu] \le u \le C(\mathbb{K}_\mu[\varrho\nu] + \delta^\alpha) \quad \textit{a.e. in } \Omega. \tag{3.22}$$

Proof. By using an argument as in the proof of Proposition 3.4, Proposition 3.5 and Proposition 3.6, we obtained the desired results. □

The above results allow to study elliptic equations with interior and boundary measures.

Proposition 3.14. *Let $p > 0$, $\sigma > 0$, $\varrho > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$ and $\nu \in \mathfrak{M}^+(\partial\Omega)$. If* (1.18) *and* (1.19) *hold then problem* (1.17) *admits a weak solution u satisfying* (1.20) *for $\sigma > 0$ and $\varrho > 0$ small enough if $p > 1$, for any $\sigma > 0$ and $\varrho > 0$ if $0 < p < 1$.*

Furthermore if $0 < p < p_\mu$ there exists a constant $C > 0$ such that for any weak solution u of (1.17) *estimate* (1.21) *holds.*

Proof. We adapt the argument in the proof of [4, Theorem 3.13]. Put $v := u - \mathbb{K}_\mu[\varrho\nu]$ then v satisfies

$$\begin{cases} -L_\mu v = (v + \mathbb{K}_\mu[\varrho\nu])^p + \sigma\tau & \text{in } \Omega, \\ \mathrm{tr}(v) = 0. \end{cases} \tag{3.23}$$

Consider the following problem

$$\begin{cases} -L_\mu w = c_p w^p + c_p(\mathbb{K}_\mu[\varrho\nu])^p + \sigma\tau & \text{in } \Omega, \\ \mathrm{tr}(w) = 0 \end{cases} \tag{3.24}$$

where $c_p := \max\{1, 2^{p-1}\}$. Since (1.18) holds, it follows that $\mathbb{K}_\mu[\nu]^p \in L^1(\Omega; \delta^\alpha)$. Since (1.19) holds, we infer from Proposition 3.4 that problem (3.24) admits a weak solution w for $\sigma > 0$ and $\varrho > 0$ small enough if $p > 1$, for any $\sigma > 0$ and $\varrho > 0$ if $0 < p < 1$. Notice that w is a supersolution of (3.24), we infer that there is a weak solution v of (3.23) satisfying $v \le w$ a.e. in Ω. By Proposition 3.4 and (1.18), we get

$$w \le c\mathbb{G}_\mu[\mathbb{K}_\mu[\varrho\nu]^p + \sigma\tau] \le c'(\mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu]) \quad \text{a.e. in } \Omega.$$

This implies (1.20).

If $0 < p < p_\mu$ then (1.21) follows from Proposition 3.6 and Proposition 3.13 (iii). □

Proof of Theorem B. Statements (i) and (ii) follow from Lemma 3.12 and Lemma 3.3 respectively. Statement (iii) follows from Proposition 3.14. Statement (iv) follows from Proposition 3.5 and Proposition 3.13 (ii). Statement (v) is derived from Proposition 3.14 (ii). □

Proof of Theorem C. The implications (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iii) follow from Proposition 3.11, Proposition 3.9 and Proposition 3.14. We will show that (iii) $\Longrightarrow$ (ii). Since (1.17) has a weak solution for $\sigma > 0$ small and $\varrho > 0$ small, it follows that (3.5) admits a solution for $\sigma > 0$ small and (3.19) admits a solution for $\varrho > 0$ small. Due to Proposition 3.11 and Proposition 3.9, we derive (1.19) and (1.18). This completes the proof. □

4. Elliptic systems: the power case

Let $\mu \in (0, \frac{1}{4}]$. In this section, we deal with system (1.32). We recall that p_μ is defined in (1.31) and

$$q := \tilde{p}\frac{p+1}{\tilde{p}+1}, \quad \tilde{q} := p\frac{\tilde{p}+1}{p+1}.$$

Without loss of generality, we can assume that $0 < p \le \tilde{p}$. Then $p \le q \le \tilde{q} \le \tilde{p}$ if $p\tilde{p} \ge 1$. Put

$$t_\mu := \tilde{p}\left(p - p_\mu + 1\right).$$

Notice that if $q < p_\mu$ then $t_\mu < q < p_\mu$.

Lemma 4.1. *Let $p > 0$, $\tilde{p} > 0$ and $\tau \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. Assume $q < p_\mu$. Then for any $t \in (\max(0, t_\mu), \tilde{p}]$, there exists a positive constant $c = c(N, p, \tilde{p}, \mu, t, \tau)$ (independent of τ if $p > 1$) such that*

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p]^{\tilde{p}} \le c\mathbb{G}_\mu[\tau]^t. \tag{4.1}$$

In particular,

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p]^{\tilde{p}} \le C\mathbb{G}_\mu[\tau]^q, \tag{4.2}$$

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p]^{\tilde{p}}] \le C\mathbb{G}_\mu[\tau] \tag{4.3}$$

where $C = C(N, p, \tilde{p}, \mu, \tau)$.

Proof. Since $q < p_\mu$, it follows that $p < p_\mu$, hence $\max(0, p - p_\mu + 1) < 1$. Let $t \in (\max(0, t_\mu, \tilde{p}]$ then $\max(0, p - p_\mu + 1) < \frac{t}{\tilde{p}} \leq 1$. By applying Lemma 3.7 with s replaced by $\frac{t}{\tilde{p}}$ respectively in order to obtain

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p] \leq c\mathbb{G}_\mu[\tau]^{\frac{t}{\tilde{p}}},$$

which implies (4.1). Since $t_\mu < q \leq \tilde{p}$, by taking $t = q$ in (4.1) we obtain (4.2). Next, since $q < p_\mu$, by apply Lemma 3.7 with γ replaced by $\tilde{\gamma}$ and (4.2), we get

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^p]^{\tilde{p}}] \leq C\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^q] \leq C\mathbb{G}_\mu[\tau]. \quad \square$$

Lemma 4.2. *Let $p > 0$, $\tilde{p} > 0$, $\tau, \tilde{\tau} \in \mathfrak{M}^+(\Omega; \delta^\alpha)$ and $\nu, \tilde{\nu} \in \mathfrak{M}^+(\partial\Omega)$. Assume that there exist positive functions $U \in L^{\tilde{p}}(\Omega; \delta^\alpha)$ and $V \in L^p(\Omega; \delta^\alpha)$ such that*

$$\begin{aligned} U &\geq \mathbb{G}_\mu[(V + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}])^p] + \mathbb{G}_\mu[\sigma\tau], \\ V &\geq \mathbb{G}_\mu[(U + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] \end{aligned} \tag{4.4}$$

in Ω. Then there exists a weak solution (u, v) of (1.32) such that

$$\begin{aligned} \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu] &\leq u \leq U, \\ \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}] &\leq v \leq V. \end{aligned} \tag{4.5}$$

Proof. Put $u_0 := 0$ and

$$\begin{cases} v_{n+1} := \mathbb{G}_\mu[u_n^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}], & n \geq 0, \\ u_n := \mathbb{G}_\mu[v_n^p] + \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu], & n \geq 1. \end{cases} \tag{4.6}$$

We see that $0 \leq v_1 = \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}] \leq V$. It is easy to see that $\{u_n\}$ and $\{v_n\}$ are nondecreasing sequences, $0 \leq u_n \leq U$ and $0 \leq v_n \leq V$ in Ω. By monotone convergence theorem, there exist $u \in L^{\tilde{p}}(\Omega; \delta^\alpha)$ and $v \in L^p(\Omega; \delta^\alpha)$ such that $u_n \to u$ in $L^1(\Omega)$, $v_n \to v$ in $L^1(\Omega)$, $u_n^{\tilde{p}} \to u^{\tilde{p}}$ in $L^1(\Omega; \delta^\alpha)$, $v_n^p \to v^p$ in $L^1(\Omega; \delta^\alpha)$. Moreover $u \leq U$ and $v \leq V$ in Ω. By letting $n \to \infty$ in (4.6), we obtain

$$\begin{cases} v = \mathbb{G}_\mu[u^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}], \\ u = \mathbb{G}_\mu[v^p] + \mathbb{G}_\mu[\sigma\tau] + \mathbb{K}_\mu[\varrho\nu]. \end{cases} \tag{4.7}$$

Thus (u, v) is a weak solution of (1.32) and satisfies (4.5). $\square$

Proof of Theorem E. We first show that the following system has weak a solution

$$\begin{cases} -L_\mu w = (\tilde{w} + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}])^p + \sigma\tau & \text{in } \Omega, \\ -L_\mu \tilde{w} = (w + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau} & \text{in } \Omega, \\ \operatorname{tr}(u) = \operatorname{tr}(v) = 0. \end{cases} \tag{4.8}$$

Fix $\vartheta_i > 0$, $(i = 1, 2, 3, 4)$ and set

$$\Psi := \mathbb{G}_\mu[\vartheta_1 \tau + \mathbb{K}_\mu[\vartheta_2 \tilde{\nu}]^p]^{\tilde{p}} + \mathbb{K}_\mu[\vartheta_3 \nu]^{\tilde{p}} + \vartheta_4 \tilde{\tau}.$$

For $\kappa \in (0, 1]$, put

$$\sigma := \kappa^{\frac{1}{\tilde{p}}} \vartheta_1, \quad \tilde{\sigma} := \kappa \vartheta_4, \quad \varrho := \kappa^{\frac{1}{\tilde{p}}} \vartheta_3, \quad \tilde{\varrho} := \kappa^{\frac{1}{p\tilde{p}}} \vartheta_2.$$

Then from the assumption, we deduce that $\Psi \in \mathfrak{M}^+(\Omega; \delta^\alpha)$. By Lemma 4.1,

$$\mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{G}_\mu[\Psi]^p]^{\tilde{p}}] \le C\, \mathbb{G}_\mu[\Psi] \tag{4.9}$$

where $C = C(N, p, \tilde{p}, \mu, \sigma, \tilde{\sigma}, \kappa, \tau, \tilde{\tau})$. Set

$$V := A\mathbb{G}_\mu[\kappa \Psi] \quad \text{and} \quad U := \mathbb{G}_\mu[(V + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}])^p + \sigma\tau]$$

where $A > 0$ will be determined later on. We have

$$\begin{aligned}
&(U + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau} \\
&\le c \left\{ \mathbb{G}_\mu\left[(a_3 \kappa \mathbb{G}_\mu[\Psi] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}])^p \right]^{\tilde{p}} + \mathbb{G}_\mu[\sigma\tau]^{\tilde{p}} + \mathbb{K}_\mu[\varrho\nu]^{\tilde{p}} \right\} + \tilde{\sigma}\tilde{\tau} \\
&\le c \left\{ A^{p\tilde{p}} \kappa^{p\tilde{p}} \mathbb{G}_\mu[\mathbb{G}_\mu[\Psi]^p]^{\tilde{p}} + \mathbb{G}_\mu[\mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}]^p]^{\tilde{p}} \right\} + c\, \mathbb{G}_\mu[\sigma\tau]^{\tilde{p}} + c\, \mathbb{K}_\mu[\varrho\nu]^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}
\end{aligned}$$

where $c = c(p, \tilde{p})$. It follows that

$$\mathbb{G}_\mu[(U + \mathbb{K}_\mu[\varrho\nu])^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}] \le I_1 + I_2 \tag{4.10}$$

where

$$\begin{aligned}
I_1 &:= c\, A^{p\tilde{p}} \kappa^{p\tilde{p}} \mathbb{G}_\mu\left[\mathbb{G}_\mu[\mathbb{G}_\mu[\Psi]^p]^{\tilde{p}} \right] + c\, \mathbb{G}_\mu\left[\mathbb{G}_\mu[\mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}]^p]^{\tilde{p}} \right], \\
I_2 &:= c\, \mathbb{G}_\mu[\mathbb{G}_\mu[\sigma\tau]^{\tilde{p}}] + c\, \mathbb{G}_\mu[\mathbb{K}_\mu[\varrho\nu]^{\tilde{p}}] + \mathbb{G}_\mu[\tilde{\sigma}\tilde{\tau}].
\end{aligned}$$

We first estimate I_1. Observe that

$$\mathbb{G}_\mu\left[\mathbb{G}_\mu[\mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}]^p]^{\tilde{p}} \right] = \mathbb{G}_\mu\left[\mathbb{G}_\mu[\mathbb{K}_\mu[\vartheta_2\tilde{\nu}]^p]^{\tilde{p}} \right] \le \mathbb{G}_\mu[\kappa\Psi].$$

This, together with (4.9) implies

$$I_1 \le c\, (A^{p\tilde{p}} \kappa^{p\tilde{p}-1} + 1) \mathbb{G}_\mu[\kappa\Psi]. \tag{4.11}$$

Next it is easy to see that

$$I_2 \le c\, \mathbb{G}_\mu[\kappa\Psi]. \tag{4.12}$$

By collecting (4.10), (4.11) and (4.12), we obtain

$$\mathbb{G}_\mu[(U+\mathbb{K}_\mu[\varrho\nu])^{\tilde p}+\tilde\sigma\tilde\tau]\le c(A^{p\tilde p}\kappa^{p\tilde p-1}+1)\mathbb{G}_\mu[\kappa\Psi] \tag{4.13}$$

with another constant c. We will choose A and κ such that

$$c\,(A^{p\tilde p}\kappa^{p\tilde p-1}+1)\le A. \tag{4.14}$$

If $p\tilde p>1$ then we can choose $A>0$ large enough and then choose $\kappa>0$ small enough (depending on A) such that (4.14) holds. If $p\tilde p<1$ then for any $\kappa>0$ there exists A large enough such that (4.14) holds. For such A and κ, we obtain

$$\mathbb{G}_\mu[(U+\mathbb{K}_\mu[\varrho\nu])^{\tilde p}+\tilde\sigma\tilde\tau]\le V.$$

By Lemma 4.2, there exists a weak solution $(w,\tilde w)$ of (4.8) for $\sigma>0$, $\tilde\sigma>0$, $\nu>0$, $\tilde\nu>0$ small if $p\tilde p>1$, for any $\sigma>0$, $\tilde\sigma>0$, $\nu>0$, $\tilde\nu>0$ if $p\tilde p<1$. Moreover, $(w,\tilde w)$ satisfies

$$\tilde w\approx\mathbb{G}_\mu[\omega], \tag{4.15}$$

$$w\approx\mathbb{G}_\mu[(\mathbb{G}_\mu[\omega]+\mathbb{K}_\mu[\tilde\nu])^p]+\mathbb{G}_\mu[\tau] \tag{4.16}$$

where $C=C(N,p,\tilde p,\mu,\Omega,\sigma,\tilde\sigma,\tau,\tilde\tau)$.

Next put $u:=w+\mathbb{K}_\mu[\varrho\nu]$ and $v:=\tilde w+\mathbb{K}_\mu[\tilde\varrho\tilde\nu]$ then (u,v) is a weak solution of (1.32). Moreover (1.33) and (1.34) follow directly from (4.15) and (4.16). □

Proof of Theorem F. Put $\tau^*:=\max\{\tau,\tilde\tau\}$ and $\nu^*:=\max\{\nu,\tilde\nu\}$. Fix $\vartheta>0$, $\tilde\vartheta>0$ and for $\kappa\in(0,1]$, put $\sigma=\varrho=(\kappa\vartheta)^{\frac{1}{\tilde p}}$ and $\tilde\sigma=\kappa\tilde\vartheta$, $\tilde\varrho=(\kappa\tilde\vartheta)^{\frac{1}{p\tilde p}}$. Set

$$\tau^\#:=\vartheta\tau+\tilde\vartheta\tilde\tau\quad\text{and}\quad\nu^\#:=\vartheta\nu+\tilde\vartheta\tilde\nu$$

then $\tau^\#\le(\vartheta+\tilde\vartheta)\tau^*$ and $\nu^\#\le(\vartheta+\tilde\vartheta)\nu^*$.

Put $V:=A(\mathbb{G}_\mu[\kappa\tau^\#]+\mathbb{K}_\mu[\kappa\nu^\#])$ where $A>0$ will be determined later on and put $U:=\mathbb{G}_\mu[(V+\mathbb{K}_\mu[\tilde\varrho\tilde\nu])^p+\sigma\tau]$.

We have

$$\begin{aligned}U^{\tilde p}+\tilde\sigma\tilde\tau\le{}& c\,A^{p\tilde p}\kappa^{p\tilde p}\Big\{\mathbb{G}_\mu[\mathbb{G}_\mu[\tau^\#]^p]^{\tilde p}+\mathbb{G}_\mu[\mathbb{K}_\mu[\nu^\#]^p]^{\tilde p}\Big\}+c\,\sigma^{\tilde p}\mathbb{G}_\mu[\tau]^{\tilde p}\\&+c\,\varrho^{\tilde p}\mathbb{K}_\mu[\nu]^{\tilde p}+c\,\tilde\sigma\tilde\tau,\end{aligned}$$

with $c=c(p,\tilde p)$. It follows that

$$\mathbb{G}_\mu[(U+\mathbb{K}_\mu[\varrho\nu])^{\tilde p}+\tilde\sigma\tilde\tau]\le c\,(J_1+J_2) \tag{4.17}$$

where

$$J_1:=A^{p\tilde p}\kappa^{p\tilde p}\Big\{\mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{G}_\mu[\tau^\#]^p]^{\tilde p}]+\mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{K}_\mu[\nu^\#]^p]^{\tilde p}]\Big\},$$

$$J_2 := \sigma^{\tilde{p}}\mathbb{G}_\mu[\mathbb{G}_\mu[\tau]^{\tilde{p}}] + \varrho^{\tilde{p}}\mathbb{G}_\mu[\mathbb{K}_\mu[\nu]^{\tilde{p}}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}] + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}].$$

We first estimate J_1. We have

$$J_1 \le A^{p\tilde{p}}\kappa^{p\tilde{p}}(\vartheta+\tilde{\vartheta})^{p\tilde{p}}\Big\{\mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{G}_\mu[\tau^*]^p]^{\tilde{p}}] + \mathbb{G}_\mu[\mathbb{G}_\mu[\mathbb{K}_\mu[\nu^*]^p]^{\tilde{p}}]\Big\}.$$

By (1.35), (1.36) and Proposition 3.11, Proposition 3.9 we infer that

$$J_1 \le c\, A^{p\tilde{p}}\kappa^{p\tilde{p}}(\vartheta+\tilde{\vartheta})^{p\tilde{p}}(\mathbb{G}_\mu[\tau^*] + \mathbb{K}_\mu[\nu^*])$$

where c is a positive constant. Therefore

$$J_1 \le c\, A^{p\tilde{p}}\kappa^{p\tilde{p}}(\vartheta+\tilde{\vartheta})^{p\tilde{p}}\max(\vartheta^{-1},\tilde{\vartheta}^{-1})(\mathbb{G}_\mu[\tau^\#] + \mathbb{K}_\mu[\nu^\#]). \tag{4.18}$$

We next estimate J_2. Again by (1.35), (1.36) and Proposition 3.11, Proposition 3.9, we deduce

$$\begin{aligned} J_2 &\le c\,(\sigma^{\tilde{p}}\mathbb{G}_\mu[\tau] + \varrho^{\tilde{p}}\mathbb{K}_\mu[\nu] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}] + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]) \\ &= c\,\kappa(\mathbb{G}_\mu[\tau^\#] + \mathbb{K}_\mu[\nu^\#]). \end{aligned} \tag{4.19}$$

Combining (4.17), (4.18) and (4.19) implies

$$\mathbb{G}_\mu[U^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}] \le C(A^{p\tilde{p}}\kappa^{p\tilde{p}-1}+1)(\mathbb{G}_\mu[\kappa\tau^\#] + \mathbb{K}_\mu[\kappa\nu^\#]) \tag{4.20}$$

where C is another positive constant. We choose $A>0$ and $\kappa>0$ such that

$$C(A^{p\tilde{p}}\kappa^{p\tilde{p}-1}+1) \le A. \tag{4.21}$$

Since $p\tilde{p}>1$, one can choose A large enough and then choose $\kappa>0$ small enough such that (4.21) holds. For such A and κ, we have

$$\mathbb{G}_\mu[U^{\tilde{p}} + \tilde{\sigma}\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\varrho}\tilde{\nu}] \le V.$$

By Lemma 4.2, there exists a weak solution (u,v) of (1.32) which satisfies (1.37). □

5. General nonlinearities

5.1. Absorption case

In this section we treat system (1.38) with $\epsilon=-1$. We recall that Λ_g and $\Lambda_{\tilde{g}}$ are defined in (1.39).

Proof of Theorem G. Step 1: We claim that

$$\int_\Omega g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\sigma}|])\delta^\alpha dx + \int_\Omega \tilde{g}(\mathbb{K}_\mu[|\nu|] + \mathbb{G}_\mu[|\tau|])\delta^\alpha dx < \infty. \tag{5.1}$$

For $\lambda > 0$, set $\tilde{A}_\lambda := \{x \in \Omega : \mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|] > \lambda\}$ and $a(\lambda) := \int_{\tilde{A}_\lambda} \delta^\alpha dx$. We write

$$\begin{aligned}\left\| g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|]) \right\|_{L^1(\Omega;\delta^\alpha)} &= \int_{\tilde{A}_1} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx \\ &\quad + \int_{\tilde{A}_1^c} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx \\ &\leq \int_{\tilde{A}_1} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx + g(1) \int_\Omega \delta^\alpha dx.\end{aligned} \tag{5.2}$$

We have

$$\int_{\tilde{A}_1} g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx = a(1)g(1) + \int_1^\infty a(s)dg(s).$$

On the other hand, by (2.2) and Proposition 2.4 one gets, for every $s > 0$,

$$a(s) \leq C \left(\left\| \mathbb{K}_\mu[|\tilde{\nu}|] \right\|^{p_\mu}_{L_w^{p_\mu}(\Omega;\delta^\alpha)} + \left\| \mathbb{G}_\mu[|\tilde{\tau}|] \right\|^{p_\mu}_{L_w^{p_\mu}(\Omega;\delta^\alpha)} \right) s^{-p_\mu} \leq C\, s^{-p_\mu} \tag{5.3}$$

where $C = C(N, \mu, \Omega, \gamma, \|\tilde{\nu}\|_{\mathfrak{M}(\partial\Omega)}, \|\tilde{\tau}\|_{\mathfrak{M}(\Omega;\delta^\alpha)})$. Thus

$$a(1)g(1) + \int_1^\infty a(s)dg(s) \leq C + C \int_1^\infty s^{-1-p_\mu} g(s)ds \leq C p_\mu \Lambda_g. \tag{5.4}$$

By combining the above estimates we obtain

$$\left\| g(\mathbb{K}_\mu[|\tilde{\nu}|] + \mathbb{G}_\mu[|\tilde{\tau}|]) \right\|_{L^1(\Omega;\delta^\alpha)} \leq C p_\mu \Lambda_g + g(1) \int_\Omega \delta^\alpha dx \leq C.$$

Similarly,

$$\left\| \tilde{g}(\mathbb{K}_\mu[|\nu|] + \mathbb{G}_\mu[|\tau|]) \right\|_{L^1(\Omega;\delta^\alpha)} \leq \tilde{C} p_\mu \Lambda_{\tilde{g}} + \tilde{g}(1) \int_\Omega \delta^\alpha dx \leq \tilde{C}.$$

Thus (5.1) follows directly.

Step 2: Existence.

Put $u_0 := \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau]$. Let v_0 be the unique weak solution of the following problem

$$\begin{cases} -L_\mu v_0 + \tilde{g}(u_0) = \tilde{\tau} & \text{in } \Omega, \\ \operatorname{tr}(v_0) = \tilde{\nu}. \end{cases}$$

For any $k \geq 1$, since $g, \tilde{g}$ satisfy (5.1) there exist functions u_k and v_k satisfying

$$\begin{cases} -L_\mu u_k + g(v_{k-1}) = \tau & \text{in } \Omega, \\ -L_\mu v_k + \tilde{g}(u_k) = \tilde{\tau} & \text{in } \Omega, \\ \operatorname{tr}(u_k) = \nu, \quad \operatorname{tr}(v_k) = \tilde{\nu}. \end{cases} \tag{5.5}$$

Moreover

$$\begin{aligned} u_k + \mathbb{G}_\mu[g(v_{k-1})] &= \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu], \\ v_k + \mathbb{G}_\mu[\tilde{g}(u_k)] &= \mathbb{G}_\mu[\tilde{\tau}] + \mathbb{K}_\mu[\tilde{\nu}]. \end{aligned} \tag{5.6}$$

Since $g, \tilde{g} \geq 0$, it follows that, for every $k \geq 1$,

$$\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[g(\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}])] \leq u_k \leq \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] = u_0$$

and

$$\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\tilde{g}(\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau])] \leq v_k \leq \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}]$$

in Ω. Now, suppose that for some $k \geq 1$, $u_k \leq u_{k-1}$. Since g and $\tilde{g}$ are nondecreasing, we deduce that

$$\begin{aligned} v_k &= \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\tilde{g}(u_k)] \geq \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\tilde{g}(u_{k-1})] = v_{k-1}, \\ u_{k+1} &= \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[g(v_k)] \leq \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[g(v_{k-1})] = u_k. \end{aligned} \tag{5.7}$$

This means that $\{v_k\}$ is nondecreasing and $\{u_k\}$ is nonincreasing. Hence, there exist u and v such that $u_k \downarrow u$ and $v_k \uparrow v$ in Ω and

$$\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau] - \mathbb{G}_\mu[g(\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}])] \leq u \leq \mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau],$$

$$\mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}] - \mathbb{G}_\mu[\tilde{g}(\mathbb{K}_\mu[\nu] + \mathbb{G}_\mu[\tau])] \leq v \leq \mathbb{K}_\mu[\tilde{\nu}] + \mathbb{G}_\mu[\tilde{\tau}].$$

Since g and $\tilde{g}$ are continuous and nondecreasing, we infer from monotone convergence theorem and (5.1) that $g(v_k) \to g(v)$ in $L^1(\Omega; \delta^\alpha)$ and $\tilde{g}(u_k) \to \tilde{g}(u)$ in $L^1(\Omega; \delta^\alpha)$. As a consequence,

$$\begin{aligned} \mathbb{G}_\mu[\tilde{g}(u_k)] &\to \mathbb{G}_\mu[\tilde{g}(u)] \quad \text{a.e. in } \Omega, \\ \mathbb{G}_\mu[g(v_k)] &\to \mathbb{G}_\mu[g(v)] \quad \text{a.e. in } \Omega. \end{aligned}$$

By letting $k \to \infty$ in (5.6), we obtain the desired result. $\square$

5.2. Source case: subcriticality

In this section for simplicity we consider system (1.38) with $\epsilon = 1$. Assume that $g(0) = \tilde{g}(0) = 0$. In preparation for proving Theorem H, we establish the following lemma:

Lemma 5.1. *Assume $\epsilon = 1$, g and $\tilde{g}$ are bounded, nondecreasing and continuous functions in $\mathbb{R}$. Let τ, $\tilde{\tau} \in \mathfrak{M}(\Omega; \delta^\alpha)$ and $\nu, \tilde{\nu} \in \mathfrak{M}(\partial\Omega)$. Assume there exist $a_1 > 0$, $b_1 > 0$ and $q_1 > 1$ such that (1.40) and (1.41) are satisfied. Then there exist λ_*, $\tilde{\lambda}_*$, $b_* > 0$ and $\varrho_* > 0$ depending on N, μ, Ω γ, $\tilde{\gamma}$, Λ_g, $\Lambda_{\tilde{g}}$, a_1, q_1 such that the following holds. For every $b_1 \in (0, b_*)$ and $\sigma, \tilde{\sigma}, \tilde{\varrho}, \varrho \in (0, \varrho_*)$ the system*

$$\begin{cases} -L_\mu u = g(v + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]) & \text{in } \Omega, \\ -L_\mu v = \tilde{g}(u + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]) & \text{in } \Omega, \\ \operatorname{tr}(u) = \operatorname{tr}(v) = 0 \end{cases} \tag{5.8}$$

admits a weak solution (u, v) satisfying

$$\begin{aligned} &\|u\|_{L_w^{p_\mu}(\Omega;\delta^\alpha)} + \|u\|_{L^{q_1}(\Omega;\delta^{\alpha-1})} \le \lambda_*, \\ &\|v\|_{L_w^{p_\mu}(\Omega;\delta^\alpha)} + \|v\|_{L^{q_1}(\Omega;\delta^{\alpha-1})} \le \tilde{\lambda}_*. \end{aligned} \tag{5.9}$$

Proof. Without loss of generality, we assume that $\|\tau\|_{\mathfrak{M}(\Omega;\delta^\alpha)} = \|\tilde{\tau}\|_{\mathfrak{M}(\Omega;\delta^\alpha)} = \|\nu\|_{\mathfrak{M}(\partial\Omega)} = \|\tilde{\nu}\|_{\mathfrak{M}(\partial\Omega)} = 1$. We shall use Schauder fixed point theorem to show the existence of positive weak solutions of (5.8). Define

$$\begin{aligned} &\mathbb{S}(w) := \mathbb{G}_\mu[g(w + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])], \\ &\tilde{\mathbb{S}}(w) := \mathbb{G}_\mu[\tilde{g}(w + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau])], \quad \forall w \in L^1(\Omega). \end{aligned} \tag{5.10}$$

Set

$$\begin{aligned} &\mathbf{M}_1(w) := \|w\|_{L_w^{p_\mu}(\Omega;\delta^\alpha)}, \quad \forall w \in L_w^{p_\mu}(\Omega; \delta^\alpha), \\ &\tilde{\mathbf{M}}_1(w) := \|w\|_{L_w^{p_\mu}(\Omega;\delta^\alpha)}, \quad \forall w \in L_w^{p_\mu}(\Omega; \delta^\alpha), \\ &\mathbf{M}_2(w) := \|w\|_{L^{q_1}(\Omega;\delta^{\alpha-1})}, \quad \forall w \in L^{q_1}(\Omega; \delta^{\alpha-1}), \\ &\mathbf{M}(w) := \mathbf{M}_1(w) + \mathbf{M}_2(w), \quad \forall w \in L_w^{p_\mu}(\Omega; \delta^\alpha) \cap L^{q_1}(\Omega; \delta^{\alpha-1}), \\ &\tilde{\mathbf{M}}(w) := \tilde{\mathbf{M}}_1(w) + \mathbf{M}_2(w), \quad \forall w \in L_w^{p_\mu}(\Omega; \delta^\alpha) \cap L^{q_1}(\Omega; \delta^{\alpha-1}). \end{aligned}$$

Step 1: Upper bound for $g(w + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])$ in $L^1(\Omega; \delta^\alpha)$ with $w \in L_w^{p_\mu}(\Omega; \delta^\alpha) \cap L^{q_1}(\Omega; \delta^{\alpha-1})$.

For $\lambda > 0$, set $\tilde{B}_\lambda := \{x \in \Omega : |w| + \tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|] + \tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|] > \lambda\}$ and $b(\lambda) := \int_{\tilde{B}_\lambda} \delta^\alpha dx$. We write

$$\begin{aligned}\left\| g(w+\tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]+\tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\right\|_{L^1(\Omega;\delta^\alpha)} \leq & \int\limits_{\tilde{B}_1} g(|w|+\tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|]+\tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx \\ & + \int\limits_{\tilde{B}_1^c} g(|w|+\tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|]+\tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx \\ & - \int\limits_{\tilde{B}_1} g(-|w|-\tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|]-\tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx \\ & - \int\limits_{\tilde{B}_1^c} g(-|w|-\tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|]-\tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|])\delta^\alpha dx \\ & =: I+II+III+IV. \end{aligned} \tag{5.11}$$

We first estimate I. Since $g \in C(\mathbb{R}_+)$ is nondecreasing, one gets

$$I = b(1)g(1) + \int\limits_1^\infty b(s)dg(s).$$

Since g is bounded, there exists an increasing sequence of real positive number $\{\ell_j\}$ such that

$$\lim_{j\to\infty} \ell_j = \infty \quad \text{and} \quad \lim_{j\to\infty} \ell_j^{-p_\mu} g(\ell_j) = 0. \tag{5.12}$$

Observe that

$$\int\limits_1^\infty b(s)dg(s) = \lim_{j\to\infty} \int\limits_\lambda^{\ell_j} b(s)dg(s).$$

On the other hand, by (2.2) one gets, for every $s>0$,

$$a(s) \leq \left\| |w|+\tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|]+\tilde{\sigma}\mathbb{G}_\mu[|\tilde{\tau}|]\right\|^{p_\mu}_{L_w^{p_\mu}(\Omega;\delta^\alpha)} s^{-p_\mu} \leq C(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu} s^{-p_\mu} \tag{5.13}$$

where $C=C(N,\mu,\Omega)$. Using (5.13), we obtain

$$\begin{aligned} & b(1)g(1) + \int\limits_1^{\ell_j} b(s)dg(s) \\ & \quad \leq C(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu} g(1) + C(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu} \int\limits_1^{\ell_j} s^{-p_\mu} dg(s) \end{aligned}$$

$$\leq C(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu}\ell_j^{-p_\mu}g(\ell_j)+Cp_\mu(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu}\int_1^{\ell_j} s^{-1-p_\mu}g(s)ds.$$

By virtue of (5.12), letting $j\to\infty$ yields

$$I \leq Cp_\mu(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu}\int_1^{\infty} s^{-1-p_\mu}g(s)ds. \tag{5.14}$$

Similarly we have

$$III \leq -Cp_\mu(\mathbf{M}_1(w)+\tilde{\varrho}+\tilde{\sigma})^{p_\mu}\int_1^{\infty} s^{-1-p_\mu}g(-s)ds.$$

To handle the remaining terms II, III, without lost of generality, we assume $q_1 \in (1, \frac{N+\alpha-1}{N+\alpha-2})$. Since g satisfies condition (1.40), it follows that

$$\begin{aligned}\max\{II, IV\} &\leq a_1\int_{\tilde{B}_1^c}(|w|+\tilde{\varrho}\mathbb{K}_\mu[|\tilde{\nu}|]+\tilde{\sigma}\mathbb{K}_\mu[|\tilde{\tau}|])^{q_1}\delta^\alpha dx+b_1\int_{\tilde{B}_1^c}\delta^\alpha dx\\ &\leq a_1C\int_\Omega |w|^{q_1}\delta^\alpha dx+a_1c_{34}(\tilde{\varrho}^{q_1}+\tilde{\sigma}^{q_1})+b_1C\\ &\leq a_1C\mathbf{M}_2(w)^{q_1}+a_1C(\tilde{\varrho}^{q_1}+\tilde{\sigma}^{q_1})+b_1C\end{aligned} \tag{5.15}$$

where $C=C(N,\mu,\Omega)$.

Combining (5.11), (5.14) and (5.15) yields

$$\left\|g(w+\tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]+\tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\right\|_{L^1(\Omega;\delta^\alpha)} \leq C\Lambda_g\mathbf{M}_1(w)^{p_\mu}+a_1C\mathbf{M}_2(w)^{q_1}+b_1C+d_{\tilde{\varrho},\tilde{\sigma}} \tag{5.16}$$

where $d_{\tilde{\varrho},\tilde{\sigma}}=C\Lambda_g(\tilde{\varrho}^{p_\mu}+\tilde{\sigma}^{p_\mu})+a_1C(\tilde{\varrho}^{q_1}+\tilde{\sigma}^{q_1})$.

Step 2: Estimates on $\mathbf{M}_1$, $\mathbf{M}_2$ and $\mathbf{M}$.

From (2.13), we have

$$\begin{aligned}\tilde{\mathbf{M}}_1(\mathbb{S}(w)) &= \left\|\mathbb{G}_\mu[g(w+\tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]+\tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\right\|_{L_w^{p_\mu}(\Omega;\delta^\alpha)}\\ &\leq C\left\|g(w+\tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]+\tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\right\|_{L^1(\Omega;\delta^\alpha)}.\end{aligned} \tag{5.17}$$

It follows that

$$\tilde{\mathbf{M}}_1(\mathbb{S}(w)) \leq C\Lambda_g\mathbf{M}_1(w)^{p_\mu}+a_1C\mathbf{M}_2(w)^{q_1}+b_1C+Cd_{\tilde{\varrho},\tilde{\sigma}}. \tag{5.18}$$

Applying (2.13), we get

$$\begin{aligned}\mathbf{M}_2(\mathbb{S}(w)) &= \left\|\mathbb{G}_\mu[g(w+\tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]+\tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\right\|_{L^{q_1}(\Omega;\delta^{\alpha-1})} \\ &\leq C\left\|g(w+\tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}]+\tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}])\right\|_{L^1(\Omega;\delta^\alpha)},\end{aligned}$$

which implies

$$\mathbf{M}_2(\mathbb{S}(w)) \leq C\Lambda_g \mathbf{M}_1(w)^{p_\mu} + a_1 C\mathbf{M}_2(w)^{q_1} + b_1 C + Cd_{\tilde{\varrho},\tilde{\sigma}}. \tag{5.19}$$

Consequently,

$$\tilde{\mathbf{M}}(\mathbb{S}(w)) \leq C\Lambda_g \mathbf{M}_1(w)^{p_\mu} + a_1 C\mathbf{M}_2(w)^{q_1} + b_1 C + Cd_{\tilde{\varrho},\tilde{\sigma}}. \tag{5.20}$$

Similarly, we can show that

$$\mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \tilde{C}\Lambda_{\tilde{g}} \tilde{\mathbf{M}}_1(w)^{p_\mu} + a_1 \tilde{C}\mathbf{M}_2(w)^{q_1} + b_1 \tilde{C} + \tilde{C}d_{\varrho,\sigma} \tag{5.21}$$

where $\tilde{C}$ is a positive constant. Define the functions η and $\tilde{\eta}$ as follows

$$\begin{aligned}\eta(\lambda) &:= \max\{C\Lambda_g, \tilde{C}\Lambda_{\tilde{g}}\}\lambda^{p_\mu} + \max\{C,\tilde{C}\}a_1\lambda^{q_1} + \max\{C,\tilde{C}\}b_1 + \max\{Cd_{\tilde{\varrho},\tilde{\sigma}}, \tilde{C}d_{\varrho,\sigma}\} \\ \tilde{\eta}(\lambda) &:= \max\{C\Lambda_g, \tilde{C}\Lambda_{\tilde{g}}\}\lambda^{p_\mu} + \max\{C,\tilde{C}\}a_1\lambda^{q_1} + \max\{C,\tilde{C}\}b_1 + \max\{Cd_{\tilde{\varrho},\tilde{\sigma}}, \tilde{C}d_{\varrho,\sigma}\}\end{aligned}$$

where C and $\tilde{C}$ are the constants in (5.20) and (5.21) respectively. By (5.20) and (5.21), we deduce

$$\tilde{\mathbf{M}}(\mathbb{S}(w)) \leq \eta(\mathbf{M}(w)) \quad \text{and} \quad \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \tilde{\eta}(\tilde{\mathbf{M}}(w)).$$

Since $p_\mu > 1$ and $q_1 > 1$, there exist $\varrho_* > 0$ and $b_* > 0$ depending on N, μ, Ω, Λ_g, $\Lambda_{\tilde{g}}$, a_1, q_1 such that for any $\varrho, \tilde{\varrho} \in (0, \varrho_*)$ and $b_1 \in (0, b_*)$ there exist $\lambda_* > 0$ and $\tilde{\lambda}_* > 0$ such that

$$\eta(\lambda_*) = \tilde{\lambda}_* \quad \text{and} \quad \tilde{\eta}(\tilde{\lambda}_*) = \lambda_*.$$

Here λ_* and $\tilde{\lambda}_*$ depend on N, μ, Ω, Λ_g, $\Lambda_{\tilde{g}}$, a_1, q_1. Therefore,

$$\begin{aligned}\mathbf{M}(w) \leq \lambda_* &\Longrightarrow \tilde{\mathbf{M}}(\mathbb{S}(w)) \leq \tilde{\lambda}_* \\ \tilde{\mathbf{M}}(w) \leq \tilde{\lambda}_* &\Longrightarrow \mathbf{M}(\tilde{\mathbb{S}}(w)) \leq \lambda_*.\end{aligned} \tag{5.22}$$

Step 3: To apply Schauder fixed point theorem.

For $w_1, w_2 \in L^1(\Omega)$, put

$$\mathbb{T}(w_1, w_2) := (\mathbb{S}(w_2), \tilde{\mathbb{S}}(w_1)), \tag{5.23}$$

$$\mathcal{D} := \{(\varphi, \tilde{\varphi}) \in L^1_+(\Omega) \times L^1_+(\Omega) : \mathbf{M}(\varphi) \leq \lambda_* \text{ and } \tilde{\mathbf{M}}(\tilde{\varphi}) \leq \tilde{\lambda}_*\}.$$

Clearly, $\mathcal{D}$ is a convex subset of $L^1(\Omega) \times L^1(\Omega)$. We shall show that $\mathcal{D}$ is a closed subset of $(L^1(\Omega))^2$. Indeed, let $\{(\varphi_m, \tilde{\varphi}_m)\}$ be a sequence in $\mathcal{D}$ converging to $(\varphi, \tilde{\varphi})$ in $(L^1(\Omega))^2$. Obviously, $\varphi \geq 0$ and $\tilde{\varphi} \geq 0$. We can extract a subsequence, still denoted by the same notation, such that $(\varphi_m, \tilde{\varphi}_m) \to (\varphi, \tilde{\varphi})$ a.e. in Ω. Consequently, by Fatou's lemma,

$$\mathbf{M}_i(\varphi) \leq \liminf_{m\to\infty} \mathbf{M}_i(\varphi_m), \quad \mathbf{M}_i(\tilde{\varphi}) \leq \liminf_{m\to\infty} \mathbf{M}_i(\tilde{\varphi}_m)$$

for $i = 1, 2$. It follows that $\mathbf{M}(\varphi) \leq \lambda_*$ and $\mathbf{M}(\tilde{\varphi}) \leq \tilde{\lambda}_*$. So $(\varphi, \tilde{\varphi}) \in \mathcal{D}$ and therefore $\mathcal{D}$ is a closed subset of $L^1(\Omega) \times L^1(\Omega)$.

Clearly, $\mathbb{T}$ is well defined in $\mathcal{D}$. For $(w, \tilde{w}) \in \mathcal{D}$, we get $\mathbf{M}(w) \leq \lambda_*$ and $\mathbf{M}(\tilde{w}) \leq \tilde{\lambda}_*$, hence $\tilde{\mathbf{M}}(\mathbb{S}(w)) \leq \tilde{\lambda}_*$ and $\mathbf{M}(\mathbb{S}(\tilde{w})) \leq \lambda_*$. It follows that $\mathbb{T}(\mathcal{D}) \subset \mathcal{D}$.

We observe that $\mathbb{T}$ is continuous. Indeed, if $w_m \to w$ and $\tilde{w}_m \to \tilde{w}$ as $m \to \infty$ in $L^1(\Omega)$ then since $g, \tilde{g} \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$, it follows that

$$g(\tilde{w}_m + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]) \to g(\tilde{w} + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]) \quad \text{in } L^1(\Omega; \delta^\alpha),$$

and

$$\tilde{g}(w_m + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]) \to \tilde{g}(w + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]) \quad \text{in } L^1(\Omega; \delta^\alpha)$$

as $m \to \infty$. By (2.13), $\mathbb{S}(\tilde{w}_m) \to \mathbb{S}(\tilde{w})$ and $\tilde{\mathbb{S}}(w_m) \to \tilde{\mathbb{S}}(w)$ as $m \to \infty$ in $L^1(\Omega)$. Thus $\mathbb{T}(w_m, \tilde{w}_m) \to \mathbb{T}(w, \tilde{w})$ in $L^1(\Omega) \times L^1(\Omega)$.

We next show that $\mathbb{T}$ is a compact operator. Let $\{(w_m, \tilde{w}_n)\} \subset \mathcal{D}$ and for each $m \geq 1$, put $\psi_m = \mathbb{S}(\tilde{w}_m)$ and $\tilde{\psi}_m = \tilde{\mathbb{S}}(w_m)$. Hence $\{\Delta\psi_m\}$ and $\{\Delta\tilde{\psi}_m\}$ are uniformly bounded in $L^p(G)$ for every subset $G \Subset \Omega$. Therefore $\{\psi_m\}$ is uniformly bounded in $W^{1,p}(G)$. Consequently, there exists a subsequence, still denoted by the same notation, and functions $\psi, \tilde{\psi}$ such that $(\psi_m, \tilde{\psi}_m) \to (\psi, \tilde{\psi})$ a.e. in Ω. By dominated convergence theorem, $(\psi_m, \tilde{\psi}_m) \to (\psi, \tilde{\psi})$ in $L^1(\Omega) \times L^1(\Omega)$. Thus $\mathbb{T}$ is compact.

By Schauder fixed point theorem there is $(u, v) \in \mathcal{D}$ such that $\mathbb{T}(u, v) = (u, v)$. □

Proof of Theorem H.I. Let $\{g_n\}$ and $\{\tilde{g}_n\}$ be the sequences of continuous, nondecreasing functions defined on $\mathbb{R}$ such that

$$\begin{aligned} &g_n(0) = g(0),\ |g_n| \leq |g_{n+1}| \leq |g|,\ \sup_{\mathbb{R}} |g_n| = n \text{ and } \lim_{n\to\infty} \|g_n - g\|_{L^\infty_{\text{loc}}(\mathbb{R})} = 0, \\ &\tilde{g}_n(0) = \tilde{g}(0),\ |\tilde{g}_n| \leq |\tilde{g}_{n+1}| \leq |\tilde{g}|,\ \sup_{\mathbb{R}} |\tilde{g}_n| = n \text{ and } \lim_{n\to\infty} \|\tilde{g}_n - \tilde{g}\|_{L^\infty_{\text{loc}}(\mathbb{R})} = 0. \end{aligned} \tag{5.24}$$

Due to Lemma 5.1, there exist $\lambda_*, \tilde{\lambda}_*, b_* > 0$ and $\varrho_* > 0$ depending on $N, \mu, \Omega, \Lambda_g, \Lambda_{\tilde{g}}, a_1, q_1$ such that for every $b_1 \in (0, b_*)$, $\tilde{\varrho}, \varrho \in (0, \varrho_*)$ and $n \geq 1$ there exists a solution $(w_n, \tilde{w}_n) \in \mathcal{D}$ of

$$\begin{cases} -L_\mu w_n = g_n(\tilde{w}_n + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]) & \text{in } \Omega, \\ -L_\mu \tilde{w}_n = \tilde{g}_n(w_n + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]) & \text{in } \Omega, \\ \operatorname{tr}(w_n) = \operatorname{tr}(\tilde{w}_n) = 0. \end{cases} \tag{5.25}$$

For each n, set $u_n = w_n + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]$ and $v_n = \tilde{w}_n + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]$. Then

$$-\int_\Omega u_n L_\mu \phi dx = \int_\Omega g_n(v_n)\phi dx + \sigma \int_\Omega \phi d\tau - \varrho \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \phi \, dx \quad \forall \phi \in \mathbf{X}_\mu(\Omega), \tag{5.26}$$

$$-\int_\Omega v_n L_\mu \phi dx = \int_\Omega \tilde{g}_n(u_n)\phi dx + \tilde{\sigma} \int_\Omega \phi d\tilde{\tau} - \tilde{\varrho} \int_\Omega \mathbb{K}_\mu[\tilde{\nu}] L_\mu \phi \, dx \quad \forall \phi \in \mathbf{X}_\mu(\Omega). \tag{5.27}$$

Since $\{(w_n, \tilde{w}_n)\} \subset \mathcal{D}$ and the fact that $\Lambda_{g_n} \le \Lambda_g$, we obtain from (5.16) that

$$\|g_n(v_n)\|_{L^1(\Omega;\delta^\alpha)} \le C\Lambda_g \lambda_*^{p_\mu} + a_1 C \lambda_*^{q_1} + b_* C + d_{\varrho *} \tag{5.28}$$

Hence the sequence $\{g(v_n)\}$ is uniformly bounded in $L^1(\Omega;\delta^\alpha)$. Since $\{(w_n, \tilde{w}_n)\} \subset \mathcal{D}$, the sequence $\{\frac{\mu}{\delta^2} w_n\}$ and $\{\frac{\mu}{\delta^2} \tilde{w}_n\}$ are uniformly bounded in $L^{q_1}(G)$ for every subset $G \Subset \Omega$. As a consequence, $\{\Delta w_n\}$ and $\{\Delta \tilde{w}_n\}$ are uniformly bounded in $L^1(G)$ for every subset $G \Subset \Omega$. By regularity results for elliptic equations, there exist subsequences, still denoted by the same notations, and functions w and $\tilde{w}$ such that $(w_n, \tilde{w}_n) \to (w, \tilde{w})$ a.e. in Ω. Therefore $(u_n, v_n) \to (u, v)$ a.e. in Ω with $u = w + \varrho\mathbb{K}_\mu[\nu] + \sigma\mathbb{G}_\mu[\tau]$ and $u = \tilde{w} + \tilde{\varrho}\mathbb{K}_\mu[\tilde{\nu}] + \tilde{\sigma}\mathbb{G}_\mu[\tilde{\tau}]$. Moreover $(\tilde{g}_n(u_n), g_n(v_n)) \to (\tilde{g}(u), g(v))$ a.e. in Ω.

We show that $u_n \to u$ in $L^1(\Omega;\delta^\alpha)$. Since $\{w_n\}$ is uniformly bounded in $L^{q_1}(\Omega;\delta^{\alpha-1})$, by (2.14), we derive that $\{u_n\}$ is uniformly bounded in $L^{q_1}(\Omega;\delta^\alpha)$. Due to Holder inequality, $\{u_n\}$ is uniformly integrable with respect to $\delta^\alpha dx$. We invoke Vitali convergence theorem to derive that $u_n \to u$ in $L^1(\Omega;\delta^\alpha)$. Similarly, one can prove that $v_n \to v$ in $L^1(\Omega;\delta^\alpha)$.

We next prove that $g_n(v_n) \to g(v)$ in $L^1(\Omega;\delta^\alpha)$. For $\lambda > 0$ and $n \in \mathbb{N}$ set $B_{n,\lambda} := \{x \in \Omega : |v_n| > \lambda\}$ and $b_n(\lambda) := \int_{B_{n,\lambda}} \delta^\alpha dx$. For any Borel set $E \subset \Omega$,

$$\begin{aligned}
\int_E g_n(v_n)\delta^\alpha dx &= \int_{E\cap B_{n,\lambda}} g_n(v_n)\delta^\alpha dx + \int_{E\cap B_{n,\lambda}^c} g_n(v_n)\delta^\alpha dx \\
&\le \int_{B_{n,\lambda}} g_n(v_n)\delta^\alpha dx + m_{g,\lambda} \int_E \delta^\alpha dx \\
&\le b_n(\lambda) g_n(\lambda) + \int_\lambda^\infty b_n(s) dg_n(s) + m_{g,\lambda} \int_E \delta^\alpha dx,
\end{aligned} \tag{5.29}$$

where $m_{g,\lambda} := \sup_{[0,\lambda]} g$. By proceeding as in the proof of Lemma 5.1 in order to get (5.14), we deduce

$$b_n(\lambda) g_n(\lambda) + \int_\lambda^\infty b_n(s) dg_n(s) \le C \int_\lambda^\infty s^{-1-p_\mu} g_n(s) ds \le C \int_\lambda^\infty s^{-1-p_\mu} g(s) ds \tag{5.30}$$

where C depends on N, μ, Λ_g, $\Lambda_{\tilde{g}}$, a_1, q_1. Note that the term on the right hand-side of (5.30) tends to 0 as $\lambda \to \infty$. Take arbitrarily $\varepsilon > 0$, there exists $\lambda > 0$ such that the right hand-side of (5.30) is smaller than $\frac{\varepsilon}{2}$. Fix such λ and put $\eta = \frac{\varepsilon}{2m_{g,\lambda}}$. Then, by (5.29),

$$\int_E \delta(x)^\alpha dx \le \eta \Longrightarrow \int_E g_n(v_n)\delta(x)^\alpha dx < \varepsilon.$$

Therefore the sequence $\{g_n(v_n)\}$ is uniformly integrable with respect to $\delta^\alpha dx$. Due to Vitali convergence theorem, we deduce that $g_n(v_n) \to g(v)$ in $L^1(\Omega; \delta^\alpha)$.

By sending $n \to \infty$ in each term of (5.26) we obtain

$$-\int_\Omega u L_\mu \phi dx = \int_\Omega g(v)\phi dx + \sigma \int_\Omega \phi \, d\tau - \varrho \int_\Omega \mathbb{K}_\mu[\nu] L_\mu \phi \, dx, \quad \forall \phi \in \mathbf{X}_\mu(\Omega). \tag{5.31}$$

Similarly, one can show that $\tilde{g}_n(u_n) \to \tilde{g}(u)$ in $L^1(\Omega; \delta^\alpha)$. By letting $n \to \infty$ in (5.27), we get

$$-\int_\Omega v L_\mu \phi dx = \int_\Omega \tilde{g}(u)\phi dx + \tilde{\sigma} \int_\Omega \phi \, d\tilde{\tau} - \tilde{\varrho} \int_\Omega \mathbb{K}_\mu[\tilde{\nu}] L_\mu \phi \, dx, \quad \forall \phi \in \mathbf{X}_\mu(\Omega). \tag{5.32}$$

Thus (u, v) is a solution of (1.27). □

5.3. Source case: sublinearity

We next deal with the case where g and $\tilde{g}$ are sublinear.

Proof of Theorem H.II. The proof is similar to that of Lemma 5.1, also based on Schauder fixed point theorem. So we point out only the main modifications. Let $\mathbb{S}$ and $\tilde{\mathbb{S}}$ be the operators defined in (5.10). Put

$$\begin{aligned} \mathbf{N}_1(w) &:= \|w\|_{L^{q_1}(\Omega;\delta^{\alpha-1})}, \quad \forall w \in L^{q_1}(\Omega; \delta^{\alpha-1}), \\ \mathbf{N}_2(w) &:= \|w\|_{L^1(\Omega;\delta^{\alpha-1})}, \quad \forall w \in L^1(\Omega; \delta^{\alpha-1}). \end{aligned}$$

Combining (2.13), (2.14) and (1.42) leads to

$$\mathbf{N}_2(\mathbb{S}(w)) \le a_2 C \mathbf{N}_1(w)^{q_1} + C(\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1} + b_2).$$

On the other hand

$$\mathbf{N}_1(\tilde{\mathbb{S}}(w)) \le a_2 C \mathbf{N}_2(w)^{q_2} + C(\varrho^{q_2} + \sigma^{q_2} + b_2)..$$

Define

$$\begin{aligned} \xi_1(\lambda) &:= a_2 C \lambda^{q_1} + C(\tilde{\varrho}^{q_1} + \tilde{\sigma}^{q_1} + b_2), \\ \xi_2(\lambda) &:= a_2 C \lambda^{q_2} + C(\varrho^{q_2} + \sigma^{q_2} + b_2). \end{aligned}$$

Then

$$\mathbf{N}_2(\mathbb{S}(w)) \le \xi_1(\mathbf{N}_1(w)) \quad \text{and} \quad \mathbf{N}_1(\tilde{\mathbb{S}}(w)) \le \xi_2(\mathbf{N}_2(w)).$$

If $q_1q_2 < 1$ then we can find λ_1 and λ_2 such that $\xi_1(\lambda_1) = \lambda_2$ and $\xi_2(\lambda_2) = \lambda_1$. Thus if $\mathbf{N}_1(w) < \lambda_1$ then $\mathbf{N}_2(\mathbb{S}(w)) < \lambda_2$ and if $\mathbf{N}_2(w) < \lambda_2$ then $\mathbf{N}_1(\tilde{\mathbb{S}}(w)) < \lambda_1$.

If $q_1q_2 = 1$ and a_2 small enough we can find, λ_1 and λ_2 such that $\xi_1(\lambda_1) = \lambda_2$ and $\xi_2(\lambda_2) = \lambda_1$.

The rest of the proof can be proceeded as in the proof of Lemma 5.1 and the proof of Theorem H.I. and we omit it. □

5.4. Source case: subcriticality and sublinearity

Proof of Theorem H.III. Set

$$\mathbf{N}(w) := \|w\|_{L^{q_1}(\Omega;\delta^{\alpha-1})}, \quad \forall w \in L^{q_1}(\Omega;\delta^{\alpha-1}).$$

By an argument similar to the proof of Lemma 5.1 and Theorem H.II, we get

$$\mathbf{N}(\mathbb{S}(w)) \le C\Lambda_g \mathbf{M}_1(w)^{p_\mu} + a_1 C\mathbf{M}_2(w)^{q_1} + b_1 C + d_{\tilde{\varrho},\tilde{\sigma}}.$$

On the other hand

$$\mathbf{M}(\tilde{\mathbb{S}}(w)) \le a_2 C\mathbf{N}(w)^{q_2} + C(\varrho^{q_2} + \sigma^{q_2} + b_2).$$

Set

$$\hat{\xi}_1(\lambda) := C\Lambda_g \lambda^{p_\mu} + a_1 C\lambda^{q_1} + b_1 C + d_{\tilde{\varrho},\tilde{\sigma}},$$
$$\hat{\xi}_2(\lambda) := a_2 C\lambda^{q_2} + C(\varrho^{q_2} + \sigma^{q_2} + b_2).$$

Then

$$\mathbf{N}(\mathbb{S}(w)) \le \hat{\xi}_1(\mathbf{M}(w)) \quad \text{and} \quad \mathbf{M}(\tilde{\mathbb{S}}(w)) \le \hat{\xi}_2(\mathbf{N}(w)).$$

We consider there cases.

Case 1: $q_1q_2 > 1$**.** Since $p_\mu > q_1$, it follows that $p_\mu q_2 > 1$. Therefore there exist $b_* > 0$ and $\varrho_* > 0$ such that for $b_1, b_2 \in (0, b_*)$ and $\varrho \in (0, \varrho_*)$ one can find $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfying

$$\hat{\xi}_1(\lambda_1) = \lambda_2 \quad \text{and} \quad \hat{\xi}_2(\lambda_2) = \lambda_1. \tag{5.33}$$

Case 2: $p_\mu q_2 = 1$**.** In this case, there exist $a_* > 0$ such that if $a_2 \in (0, a_*)$ then for every $\varrho > 0$ and $\tilde{\varrho} > 0$ one can find $\lambda_1 > 0$ and $\lambda_2 > 0$ satisfying (5.33).

Case 3: $p_\mu q_2 < 1$**.** In this case for every $\varrho > 0$ and $\tilde{\varrho} > 0$ one can find $\lambda_1 > 0$ and $\lambda_2 > 0$ such that (5.33) holds.

Hence, in any case,

$$\mathbf{M}(w) \le \lambda_1 \Longrightarrow \mathbf{N}(\mathbb{S}(w)) \le \hat{\xi}_1(\lambda_1) = \lambda_2$$
$$\mathbf{N}(w) \le \lambda_2 \Longrightarrow \mathbf{M}(\tilde{\mathbb{S}}(w)) \le \hat{\xi}_2(\lambda_2) = \lambda_1.$$

The rest of the proof can be proceeded as in the proof of Lemma 5.1 and the proof of Theorem H.II. and we omit it. □

References

[1] D.R. Adams, L.I. Heberg, Function Spaces and Potential Theory, Grundlehren der Mathematischen Wisenschaften, vol. 31, Springer-Verlag, 1999.

[2] A. Ancona, Negatively curved manifolds, elliptic operators and the Martin boundary, Ann. of Math. (2) 125 (1987) 495–536.

[3] C. Bandle, V. Moroz, W. Reichel, Boundary blowup type sub-solutions to semilinear elliptic equations with Hardy potential, J. Lond. Math. Soc. 2 (2008) 503–523.

[4] M.F. Bidaut-Véron, L. Vivier, An elliptic semilinear equation with source term involving boundary measures: the subcritical case, Rev. Mat. Iberoam. 16 (2000) 477–513.

[5] M.F. Bidaut-Véron, G. Hoang, Q.H. Nguyen, L. Véron, An elliptic semilinear equation with source term and boundary measure data: the supercritical case, J. Funct. Anal. 269 (2015) 1995–2017.

[6] M.F. Bidaut-Véron, C. Yarur, Semilinear elliptic equations and systems with measure data: existence and a priori estimates, Adv. Differential Equations 7 (2002) 257–296.

[7] S. Filippas, L. Moschini, A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrodinger operators on bounded domains, Comm. Math. Phys. 273 (2007) 237–281.

[8] M. García-Huidobro, R. Manásevich, E. Mitidieri Enzo, C.S. Yarur, Existence and nonexistence of positive singular solutions for a class of semilinear elliptic systems, Arch. Ration. Mech. Anal. 140 (3) (1997) 253–284.

[9] K.T. Gkikas, L. Véron, Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials, Nonlinear Anal. 121 (2015) 469–540.

[10] A. Gmira, L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations, Duke Math. J. 64 (1991) 271–324.

[11] N.J. Kalton, I.E. Verbitsky, Nonlinear equations and weighted norm inequality, Trans. Amer. Math. Soc. 351 (1999) 3441–3497.

[12] M. Marcus, V.J. Mizel, Y. Pinchover, On the best constant for Hardy's inequality in $\mathbb{R}^N$, Trans. Amer. Math. Soc. 350 (1998) 3237–3255.

[13] M. Marcus, V. Moroz, Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential, http://arxiv.org/abs/1603.09265.

[14] M. Marcus, P.T. Nguyen, Moderate solutions of semilinear elliptic equations with Hardy potential, Ann. Inst. H. Poincaré Anal. Non Linéaire 34 (2017) 69–88.

[15] M. Marcus, L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the subcritical case, Arch. Ration. Mech. Anal. 144 (1998) 201–231.

[16] M. Marcus, L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: the supercritical case, J. Math. Pures Appl. (9) 77 (1998) 481–524.

[17] M. Marcus, L. Véron, Nonlinear Second Order Elliptic Equations Involving Measures, De Gruyter Series in Nonlinear Analysis and Applications, 2013.

[18] P.-T. Nguyen, Elliptic equations with Hardy potential and subcritical source term, Calc. Var. Partial Differential Equations 56 (2) (2017) 44.

[19] P. Quittner, P. Souplet, Superlinear Parabolic Problems. Blow-Up, Global Existence and Steady States, Birkhäuser Verlag, Basel, 2007, xii+584 pp.

CHAPTER 5

Elliptic equations with a Hardy potential and a gradient-dependent nonlinearity

An investigation on semilinear equations with a Hardy potential and a gradient-dependent nonlinear term is presented in this chapter, which is based on a joint work with Gkikas [80]. We establish sharp existence and uniqueness results and obtain a complete description of isolated singularities in subcritical range. We also show that singularities are removable in the supercritical range.

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Elliptic Equations with Hardy Potential and Gradient-Dependent Nonlinearity

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Abstract: Let $\Omega \subset \mathbb{R}^N$ $(N \geq 3)$ be a C^2 bounded domain, and let δ be the distance to $\partial\Omega$. We study equations $(E_\pm)$, $-L_\mu u \pm g(u, |\nabla u|) = 0$ in Ω, where $L_\mu = \Delta + \frac{\mu}{\delta^2}$, $\mu \in (0, \frac{1}{4}]$ and $g\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and locally Lipschitz in its two variables with $g(0,0) = 0$. We prove that, under some subcritical growth assumption on g, equation (E_+) with boundary condition $u = \nu$ admits a solution for any nonnegative bounded measure on $\partial\Omega$, while equation (E_-) with boundary condition $u = \nu$ admits a solution provided that the total mass of ν is small. Then we analyze the model case $g(s,t) = |s|^p t^q$ and obtain a uniqueness result, which is even new with $\mu = 0$. We also describe isolated singularities of positive solutions to (E_+) and establish a removability result in terms of Bessel capacities. Various existence results are obtained for (E_-). Finally, we discuss existence, uniqueness and removability results for $(E_\pm)$ in the case $g(s,t) = |s|^p + t^q$.

Keywords: Hardy Potential, Singular Solutions, Boundary Trace, Uniqueness, Critical Exponent, Gradient Term, Isolated Singularities

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Dedicated to Laurent Véron on his 70th birthday.

1 Introduction and Main Results

Let Ω be a C^2 bounded domain in $\mathbb{R}^N$ $(N \geq 3)$, $\mu \in (0, \frac{1}{4}]$ and $\delta(x) = \delta_\Omega(x) := \operatorname{dist}(x, \partial\Omega)$. In this paper, we investigate the boundary value problem with measure data for equation

$$-L_\mu u \pm g(u, |\nabla u|) = 0 \quad \text{in } \Omega, \tag{$E_\pm$}$$

where $L_\mu = L_\mu^\Omega := \Delta + \frac{\mu}{\delta^2}$ and $g\colon \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}_+$ is nondecreasing and locally Lipschitz in its two variables with $g(0,0) = 0$. The term $\frac{\mu}{\delta^2}$ is called Hardy potential since it is related to the Hardy inequality. The nonlinearity $g(u, |\nabla u|)$ is called *absorption* (resp. *source*) if the "plus sign" (resp. "minus sign") appears in $(E_\pm)$. One prototype model to keep in mind is $g(u, |\nabla u|) = |u|^p |\nabla u|^q$.

1.1 Background and Main Contributions

The boundary value problem for $(E_\pm)$ without Hardy potential, i.e. $\mu = 0$, has received substantial attention over the last decades, starting from the pioneering work of Brezis [10]. In particular, Brezis proved that, for

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every prescribed L^1 boundary datum, the semilinear equation with absorption term

$$-\Delta u + g(u) = 0 \quad \text{in } \Omega \tag{1.1}$$

admits a unique solution. Afterwards, equation (1.1) in measure framework was first considered by Gmira and Véron in [18] where they showed that boundary value problem for (1.1) is not always solvable for every measure boundary datum. Because of its applications in many areas, equation (1.1) with $g(u, |\nabla u|) = |u|^{p-1}u$ has been intensively studied in many works, among them is the celebrated series of papers of Marcus and Véron [26–28]. These results were then extended to the equation with gradient-dependent absorption term

$$-\Delta u + g(u, |\nabla u|) = 0 \quad \text{in } \Omega.$$

We refer to [32] for the case when g depends only on ∇u and to [24, 30] for the case when g depends on both u and ∇u.

Equation (E_-) with $\mu = 0$, i.e.

$$-\Delta u - g(u, |\nabla u|) = 0 \quad \text{in } \Omega, \tag{1.2}$$

has been studied in various directions. Necessary and sufficient criteria in terms of capacities for the existence of a solution with measure boundary data were obtained in [9]. Singular solutions of (1.2) with $g(u, |\nabla u|) = |\nabla u|^q$ in a perturbation of the ball was studied in [2]. Recently, Bidaut-Véron, Garcia-Huidobro and Véron have established a priori estimates for solutions of (1.2) with $g(u, |\nabla u|) = |u|^p|\nabla u|^q$ or $g(u, |\nabla u|) = |u|^p + M|\nabla u|^q$ (see [7, 8]).

The case with Hardy potential has been intensively studied over the last decade. See e.g. Bandle, Moroz and Reichel [5], Bandle, Marcus and Moroz [4], Marcus and Nguyen [25], Gkikas and Véron [17], Marcus and Moroz [23], Nguyen [31], Gkikas and Nguyen [16]. In the aforementioned papers, the best constant in the Hardy inequality

$$C_H(\Omega) := \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla \varphi|^2\, dx}{\int_\Omega (\varphi/\delta)^2\, dx} \tag{1.3}$$

is deeply involved in the analysis. It is well known that $C_H(\Omega) \in (0, \frac{1}{4}]$ and $C_H(\Omega) = \frac{1}{4}$ if Ω is convex (see [22, Theorem 11]) or if $-\Delta\delta \geq 0$ in the sense of distributions (see [6, Theorem A]). Moreover, the infimum in (1.3) is achieved if and only if $C_H(\Omega) < \frac{1}{4}$.

Moreover, Brezis and Marcus [11, Remark 3.2] proved that, for any $\mu < \frac{1}{4}$, the eigenvalue problem

$$\lambda_\mu := \inf_{\varphi \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega (|\nabla \varphi|^2 - \frac{\mu}{\delta^2}\varphi^2)\, dx}{\int_\Omega \varphi^2\, dx} \tag{1.4}$$

admits a minimizer φ_μ in $H_0^1(\Omega)$, and hence λ_μ is the first eigenvalue of $-L_\mu$ in $H_0^1(\Omega)$. Moreover, $-L_\mu\varphi_\mu = \lambda_\mu\varphi_\mu$ in Ω. When $\mu = \frac{1}{4}$, there is no minimizer of (1.4) in $H_0^1(\Omega)$, but there exists a nonnegative function $\varphi_{\frac{1}{4}} \in H^1_{\mathrm{loc}}(\Omega)$ such that $-L_{\frac{1}{4}}\varphi_{\frac{1}{4}} = \lambda_{\frac{1}{4}}\varphi_{\frac{1}{4}}$ in Ω in the sense of distributions.

We see from (1.3) and (1.4) that $\lambda_\mu > 0$ if $\mu < C_H(\Omega)$, $\lambda_\mu = 0$ if $\mu = C_H(\Omega) < \frac{1}{4}$, while $\lambda_\mu < 0$ when $\mu > C_H(\Omega)$. It is not known if $\lambda_\mu > 0$ when $\mu = C_H(\Omega) = \frac{1}{4}$. However, if Ω is convex or if $-\Delta\delta \geq 0$ in the sense of distributions (in these cases $C_H(\Omega) = \frac{1}{4}$), then $\lambda_{\frac{1}{4}} > 0$ (see [11, Theorem II] and [6, Theorem A] with $k = 1$ and $p = 2$).

Throughout the present paper, we assume that

$$\mu \in (0, \tfrac{1}{4}] \quad \text{and} \quad \lambda_\mu > 0. \tag{1.5}$$

This assumption implies the validity of the representation theorem which states that every positive L_μ-harmonic function u in Ω (i.e. u is a solution of $L_\mu u = 0$ in Ω in the sense of distributions) can be uniquely represented in the form $u = \mathbb{K}_\mu[\nu]$ for some positive measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ (the space of positive bounded measure on $\partial\Omega$), where $\mathbb{K}_\mu$ denotes the Martin operator (see Subsection 2.2 for more details). The representation theorem is derived from Ancona [3] (see also [25, page 70]) in the case $\mu < C_H(\Omega)$ and was proved by Gkikas and Véron [17, Theorem 2.33] in the case $\mu = \frac{1}{4}$ and $\lambda_{\frac{1}{4}} > 0$.

In order to investigate the boundary behavior of L_μ-harmonic functions, Gkikas and Véron [17] introduced a notion of boundary trace in a *dynamic way* which is recalled below.

Let $D \Subset \Omega$ and $x_0 \in D$. If $h \in C(\partial D)$, then the problem

$$\begin{cases} -L_\mu u = 0 & \text{in } D, \\ u = h & \text{on } \partial D \end{cases}$$

admits a unique solution which allows to define the L_μ-harmonic measure $\omega_D^{x_0}$ on ∂D by

$$u(x_0) = \int_{\partial D} h(y)\, d\omega_D^{x_0}(y).$$

A sequence of domains $\{\Omega_n\}$ is called a *smooth exhaustion* of Ω if $\partial\Omega_n \in C^2$, $\overline{\Omega_n} \subset \Omega_{n+1}$, $\bigcup_n \Omega_n = \Omega$ and $\mathcal{H}^{N-1}(\partial\Omega_n) \to \mathcal{H}^{N-1}(\partial\Omega)$. For each n, let $\omega_{\Omega_n}^{x_0}$ be the $L_\mu^{\Omega_n}$-harmonic measure on $\partial\Omega_n$.

Definition 1.1. Let $\mu \in (0, \frac{1}{4}]$. We say that a function u possesses a *boundary trace* if there exists a measure $\nu \in \mathfrak{M}(\partial\Omega)$ (the space of bounded measure on $\partial\Omega$) such that, for any smooth exhaustion $\{\Omega_n\}$ of Ω, it holds

$$\lim_{n\to\infty} \int_{\partial\Omega_n} \phi u \, d\omega_{\Omega_n}^{x_0} = \int_{\partial\Omega} \phi \, d\nu \quad \text{for all } \phi \in C(\overline{\Omega}).$$

The boundary trace of u is denoted by $\operatorname{tr}(u)$, and we write $\operatorname{tr}(u) = \nu$.

In [17, Proposition 2.34], Gkikas and Véron proved that if $\operatorname{tr}(\mathbb{K}_\mu[\nu]) = \nu$ for every $\nu \in \mathfrak{M}(\partial\Omega)$. This fact and the representation theorem allow to characterize L_μ-harmonic functions in terms of their boundary behavior. It was shown in [15] that, when $\mu \in (0, C_H(\Omega))$, the notion of boundary trace in Definition 1.1 coincides with the notion of *normalized boundary trace* introduced by Marcus and Nguyen in [25, Definition 1.2]. This notion was employed to formulate the boundary value problem

$$\begin{cases} -L_\mu u \pm |u|^{p-1}u = 0 & \text{in } \Omega, \\ \operatorname{tr}(u) = \nu. \end{cases} \tag{1.6}$$

A complete description of the structure of positive solutions to (1.6) with "plus sign" was established in [17, 25], and various existence results for (1.6) with "minus sign" were given in [15, 31] in connection to the critical exponent

$$p_\mu := \frac{N+\alpha}{N+\alpha-2} \quad \text{with } \alpha := \frac{1}{2} + \sqrt{\frac{1}{4} - \mu}. \tag{1.7}$$

In particular, it was proved that, when $1 < p < p_\mu$, equation (1.6) with "plus sign" admits a unique solution for every finite measure $\nu \in \mathfrak{M}^+(\partial\Omega)$, while the existence phenomenon occurs for (1.6) with "minus sign" only with boundary measure of small total mass. When $p \geq p_\mu$, the nonexistence phenomenon happens, i.e. equations $(E_\pm)$ do not admit any solution with an isolated singularity. Related results were obtained in [4, 5, 23] and references therein.

Very recently, a thorough study of the boundary value problem

$$\begin{cases} -L_\mu u + |\nabla u|^q = 0 & \text{in } \Omega, \\ \operatorname{tr}(u) = \nu \end{cases} \tag{1.8}$$

was carried out in [16], revealing that the value

$$q_\mu := \frac{N+\alpha}{N+\alpha-1}$$

is a critical exponent for the solvability of (1.8). This means that if $1 < q < q_\mu$, then, for every $\nu \in \mathfrak{M}^+(\partial\Omega)$, there is a unique solution of (1.8); otherwise, if $q_\mu \leq q < 2$, singularities are removable.

Motivated by the aforementioned works, in the present paper, we aim to investigate related issues for $(E_\pm)$. Main features of a boundary value problem for $(E_\pm)$ with measure are

- the presence of the Hardy potential which blowups strongly at the boundary,
- the dependence of the nonlinearity on both solution and its gradient,
- rough data which cause the invalidity of some classical results.

The interplay between the features leads to new essential difficulties, hence complicates drastically the analysis and produces interestingly new phenomena.

Our contributions are the following.

- We establish the existence of weak solutions of $(E_\pm)$ with prescribed boundary trace ν under sharp assumption on g. In particular, by using the standard approximation method, combined with the estimates of the Green kernel and the Martin kernel as well as their gradient [15], the sub- and supersolutions theorem and the Vitali convergence theorem, we show that, for every measure $\nu \in \mathfrak{M}^+(\partial\Omega)$, equation (E_+) admits a solution. Unlike the absorption case, thanks to the Schauder fixed point theorem, we can construct a solution to $(E_\pm)$ under the smallness assumption on the total mass of the boundary datum.
- We prove the comparison principle for (E_+), which in turn implies the uniqueness. This result, which is obtained by developing the method in [16, 24] and the theory of linear equations [15, 17, 25], is new, even in the case without Hardy potential.
- We show sharp a priori estimates for singular solutions of $(E_\pm)$. This allows to study solutions with an isolated singularity. As a matter of fact, we show that there are two types of solutions with an isolated singularity of (E_+): weakly singular ones and strongly singular one. Moreover, the strongly singular solution can be obtained as the limit of the weakly singular solutions. It is interesting that this phenomenon does not occur for (E_-). The interaction of u^p and $|\nabla u|^q$ is a source of difficulties, which requires a delicate analysis and heavy computations.
- We demonstrate removability results of singularities in terms of capacities. The absorption case and source case are treated differently using different types of capacities (see [9, 16]).

Our results cover and refine most of the aforementioned works in the literature and provide a full understanding of equations with Hardy potential and gradient-dependent nonlinearity.

1.2 Main Results

First we are concerned with a boundary value problem for equations with *absorption term* of the form

$$\begin{cases} -L_\mu u + g(u, |\nabla u|) = 0 & \text{in } \Omega, \\ \operatorname{tr}(u) = \nu. \end{cases} \tag{P_+^ν}$$

Before stating the main results, let us give the definition of weak solutions of (P_+^ν).

Definition 1.2. Let $\nu \in \mathfrak{M}(\partial\Omega)$. A function u is called a *weak solution* of (P_+^ν) if

$$u \in L^1(\Omega, \delta^\alpha), \quad g(u, |\nabla u|) \in L^1(\Omega, \delta^\alpha)$$

and

$$-\int_\Omega u L_\mu \zeta \, dx + \int_\Omega g(u, |\nabla u|)\zeta \, dx = -\int_\Omega \mathbb{K}_\mu[\nu] L_\mu \zeta \, dx \quad \text{for all } \zeta \in \mathbf{X}_\mu(\Omega),$$

where the space of test function $\mathbf{X}_\mu(\Omega)$ is defined by

$$\mathbf{X}_\mu(\Omega) := \{\zeta \in H^1_{\mathrm{loc}}(\Omega) : \delta^{-\alpha}\zeta \in H^1(\Omega, \delta^{2\alpha}),\ \delta^{-\alpha} L_\mu \zeta \in L^\infty(\Omega)\}. \tag{1.9}$$

We notice that this definition is inspired by the definition in [17, Section 3.2]. For more properties of the space of test functions $\mathbf{X}_\mu(\Omega)$, we refer to [17].

Our first result is the existence of a weak solution of (P_+^ν) under an integral condition on g.

Theorem 1.3 (Existence). *Assume g satisfies*

$$\Lambda_g := \int_1^\infty g(s, s^{\frac{p_\mu}{q_\mu}}) s^{-1-p_\mu} \, ds < \infty. \tag{1.10}$$

Then, for any $\nu \in \mathfrak{M}^+(\partial\Omega)$, (P_+^ν) admits a positive weak solution $0 \le u \le \mathbb{K}_\mu[\nu]$ in Ω.

Remark 1.4. We remark the following.
(i) If $g(s,t) = |s|^p t^q$ for $s \in \mathbb{R}$, $t \in \mathbb{R}_+$, $p, q \geq 0$, $p + q > 1$, then g satisfies (1.10) if

$$(N + \alpha - 2)p + (N + \alpha - 1)q < N + \alpha. \tag{1.11}$$

(ii) If $g(s,t) = |s|^p + t^q$ for $s \in \mathbb{R}$, $t \in \mathbb{R}_+$, $p > 1$, $q > 1$, then g satisfies (1.10) if

$$1 < p < p_\mu \quad \text{and} \quad 1 < q < q_\mu. \tag{1.12}$$

It is worth noticing that this theorem is established by developing the sub- and supersolutions method in [16], in combination with the Schauder fixed point theorem and the Vitali convergence theorem.

It seems infeasible to obtain the uniqueness in case of general nonlinearity; however, when

$$g(u, |\nabla u|) = |u|^p |\nabla u|^q,$$

we are able to prove the comparison principle, which in turn implies the uniqueness. The method is delicate, relying on a regularity result (see Proposition 4.1), maximum principle (see Lemma 4.2), estimates on the gradient of subsolutions of a nonhomogeneous linear equation (see Lemma 4.4). *We emphasize that this result is new even in the case without Hardy potential, i.e.* $\mu = 0$.

Theorem 1.5 (Comparison Principle). *Assume* $g(u, |\nabla u|) = |u|^p |\nabla u|^q$ *with* $q \geq 1$ *and* p *and* q *satisfying* (1.11). *Let* $\nu_i \in \mathfrak{M}^+(\partial\Omega)$, $i = 1, 2$, *and let* u_i *be a nonnegative solution of* (P_+^ν) *with* $\nu = \nu_i$. *If* $\nu_1 \leq \nu_2$, *then* $u_1 \leq u_2$ *in* Ω.

Assume $0 \in \partial\Omega$, and denote by δ_0 the Dirac measure concentrated at 0. A complete picture of isolated singularities concentrated at 0 is depicted in the next theorem.

Theorem 1.6. *Assume* $g(u, |\nabla u|) = |u|^p |\nabla u|^q$ *with* $q \geq 1$ *and* p *and* q *satisfying* (1.11).
(I) Weak singularity. *For any* $k > 0$, *let* $u_{0,k}^\Omega$ *be the solution of*

$$\begin{cases} -L_\mu u + g(u, |\nabla u|) = 0 & \text{in } \Omega \\ \operatorname{tr}(u) = k\delta_0. \end{cases} \tag{1.13}$$

Then there exists a constant $c = c(N, \mu, \Omega) > 0$ *such that* $u_{0,k}^\Omega(x) \leq ck\delta(x)^\alpha |x|^{2-N-2\alpha}$ *for every* $x \in \Omega$ *and*

$$|\nabla u_{0,k}^\Omega(x)| \leq ck\delta(x)^{\alpha-1}|x|^{2-N-2\alpha} \quad \text{for all } x \in \Omega.$$

Moreover,

$$\lim_{\Omega \ni x \to y} \frac{u_{0,k}^\Omega(x)}{K_\mu^\Omega(x, 0)} = k. \tag{1.14}$$

Furthermore, the mapping $k \mapsto u_{0,k}^\Omega$ *is increasing.*
(II) Strong singularity. *Put* $u_{0,\infty}^\Omega := \lim_{k\to\infty} u_{0,k}^\Omega$. *Then* $u_{0,\infty}^\Omega$ *is a solution of*

$$\begin{cases} -L_\mu u + g(u, |\nabla u|) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \setminus \{0\}. \end{cases} \tag{1.15}$$

There exists a constant $c = c(N, \mu, p, q, \Omega) > 0$ *such that*

$$c^{-1}\delta(x)^\alpha |x|^{-\frac{2-q}{p+q-1}-\alpha} \leq u_{0,\infty}^\Omega(x) \leq c\delta(x)^\alpha |x|^{-\frac{2-q}{p+q-1}-\alpha} \quad \text{for all } x \in \Omega,$$
$$|\nabla u_{0,\infty}^\Omega(x)| \leq c\delta(x)^{\alpha-1}|x|^{-\frac{2-q}{p+q-1}-\alpha} \quad \text{for all } x \in \Omega.$$

Moreover,

$$\lim_{\substack{\Omega \ni x \to 0 \\ \frac{x}{|x|} = \sigma \in S_+^{N-1}}} |x|^{\frac{2-q}{p+q-1}} u_{0,\infty}^\Omega(x) = \omega(\sigma), \tag{1.16}$$

locally uniformly on the upper hemisphere $S_+^{N-1} = \mathbb{R}_+^N \cap S^{N-1}$, *where* ω *is the unique solution of problem* (4.35). *Here* $\mathbb{R}_+^N = \{x = (x_1, \dots, x_N) = (x', x_N) : x_N > 0\}$, *and* S^{N-1} *is the unit sphere in* $\mathbb{R}^N$.

Let us discuss briefly the proof of Theorem 1.6. The main ingredients in the proof of convergence (1.14) are the estimates on the Green kernel (2.1) and the Martin kernel (2.2) and condition (1.11). From the monotonicity of the sequence $\{u_{0,k}^{\Omega}\}$, universal estimate (4.15) and a standard argument, we deduce that $u_{0,\infty}^{\Omega}$ is a solution of (1.15). The proof of convergence (1.16) relies strongly on the similarity transformation T_ℓ (see (4.19)) and the study of problem (4.35) in the upper hemisphere S_+^{N-1}. The existence and uniqueness result for (4.35) is provided in Section 4.4.

When $g(u,|\nabla u|)=|u|^p|\nabla u|^q$ with $q\geq 1$, in order to deal with a wider range of p and q (i.e. p and q may not satisfy (1.11)), we make use of Bessel capacities (see Section 5). A necessary condition for the existence of a solution to (P_+^ν) and a removability result are stated in the following theorems.

Theorem 1.7 (Absolute Continuity). *Assume $g(u,|\nabla u|)=|u|^p|\nabla u|^q$ with*

$$p\geq 0,\quad 1\leq q<2,\quad p+q>1\quad \text{and}\quad (N+\alpha-2)p+(N+\alpha-1)q\geq N+\alpha.$$

Let $\nu\in\mathfrak{M}^+(\partial\Omega)$ and assume that problem (P_+^ν) has a nonnegative solution $u\in C^2(\Omega)$.
(i) *If $q\neq\alpha+1$, then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}$, i.e. $\nu(K)=0$ for any Borel set $K\subset\partial\Omega$ such that $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(K)=0$. Here $(p+q)'$ denotes the conjugate exponent of $p+q$.*
(ii) *If $q=\alpha+1$, then, for any $\varepsilon\in(0,\min\{\alpha+1,\frac{(N-1)\alpha}{\alpha+1}-(1-\alpha)\})$, ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\varepsilon}{p+\alpha+1},(p+\alpha+1)'}$. Here the capacity $C^{\mathbb{R}^{N-1}}_{s,\kappa}$ is defined in Section 5.*

Put

$$W(x):=\begin{cases}\delta(x)^{1-\alpha} & \text{if } \mu<\frac{1}{4},\\ \delta(x)^{\frac{1}{2}}|\ln\delta(x)| & \text{if } \mu=\frac{1}{4}.\end{cases}\tag{1.17}$$

We note that, by [17, Propositions 2.17, 2.18], for any $h\in C(\partial\Omega)$, there exists a unique L_μ-harmonic function $u_h\in C(\overline{\Omega})\cap L^1(\Omega,\delta^\alpha)$ such that

$$\lim_{x\in\Omega,\,x\to\xi}\frac{u_h(x)}{W(x)}=h(\xi)\quad\text{for all }\xi\in\partial\Omega.\tag{1.18}$$

In addition, $\mathrm{tr}(u_h)=h\omega^{x_0}$, where $x_0\in\Omega$ is a fixed reference point and ω^{x_0} is the L_μ-harmonic measure in Ω (see [17, Subsection 2.3] for further details). It is worth mentioning that (1.18) can be viewed as the boundary condition in the case with Hardy potential. If $\mu=0$, then $\alpha=1$ and $W(x)\equiv 1$, in which case (1.18) becomes the boundary condition in the classical sense.

The following result provides a removability result for solutions with "zero boundary condition" on $\partial\Omega\setminus K$ (see (1.20)).

Theorem 1.8 (Removability). *Assume $p\geq 0$, $1\leq q<2$, $p+q>1$ and $(N+\alpha-2)p+(N+\alpha-1)q\geq N+\alpha$. Let $K\subset\partial\Omega$ be compact such that*
(i) *$C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(K)=0$ if $q\neq\alpha+1$ or*
(ii) *$C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\varepsilon}{p+\alpha+1},(p+\alpha+1)'}(K)=0$ for some $\varepsilon\in(0,\min\{\alpha+1,\frac{(N-1)\alpha}{\alpha+1}-(1-\alpha)\})$ if $q=\alpha+1$.*
Then any nonnegative solution $u\in C^2(\Omega)\cap C(\overline{\Omega}\setminus K)$ of

$$-L_\mu u+|u|^p|\nabla u|^q=0\quad\text{in }\Omega\tag{1.19}$$

such that

$$\lim_{x\in\Omega,\,x\to\xi}\frac{u(x)}{W(x)}=0\quad\text{for all }\xi\in\partial\Omega\setminus K\tag{1.20}$$

is identically zero.

Next we deal with the boundary value problem for equations with *source term* of the form

$$\begin{cases}-L_\mu u-g(u,|\nabla u|)=0 & \text{in }\Omega,\\ \mathrm{tr}(u)=\nu.\end{cases}\tag{P_-^ν}$$

Weak solutions of (P_-^ν) are defined similarly to Definition 1.2.

Phenomena occurring in this case are different from those in the case of absorption nonlinearity. This is reflected in Theorem 1.9 which ensures the existence of a weak solution *under a smallness assumption on the total mass of the boundary data*.

In order to make the statement clear and lucid, we rewrite equation (P_-^ν) as

$$\begin{cases} -L_\mu u - g(u, |\nabla u|) = 0 & \text{in } \Omega, \\ \operatorname{tr}(u) = \varrho\nu, \end{cases} \tag{$P_-^{\varrho\nu}$}$$

where ϱ is a positive parameter and $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$.

Theorem 1.9 (Existence Result for $(P_-^{\varrho\nu})$ in Subcritical Case). *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$. Assume g satisfies* (1.10) *and*

$$g(as, bt) \le \tilde{k}(a^{\tilde{p}} + b^{\tilde{q}})g(s, t) \quad \text{for all } (a, b, s, t) \in \mathbb{R}_+^4, \tag{1.21}$$

for some $\tilde{p} > 1$, $\tilde{q} > 1$, $\tilde{k} > 0$. Then there exists $\varrho_0 > 0$ depending on $N, \mu, \Omega, \Lambda_g, \tilde{k}, \tilde{p}, \tilde{q}$ such that, for any $\varrho \in (0, \varrho_0)$, problem $(P_-^{\varrho\nu})$ admits a positive weak solution $u \ge \varrho\mathbb{K}_\mu[\nu]$ in Ω.

This result is established by combining an idea in [34] and the Schauder fixed point theorem.

Remark 1.10. It is easy to see that if $g(s, t) = |s|^p t^q$ or $g(s, t) = |s|^p + t^q$, then (1.21) holds.

The next result provides sufficient conditions for the existence of a solution to $(P_-^{\varrho\nu})$ with $g(u, |\nabla u|) = |u|^p|\nabla u|^q$ in terms of capacities. See the definition of the capacities $\mathrm{Cap}^{\partial\Omega}$ and $\mathbb{N}_{2\alpha-1,1}$ in Section 7.

Theorem 1.11 (Existence Result for $(P_-^{\varrho\nu})$). *Assume that $g(u, |\nabla u|) = |u|^p|\nabla u|^q$ with $p \ge 0$, $q \ge 0$, $p + q > 1$ and $q < \frac{1+\alpha+(1-\alpha)p}{\alpha}$. Assume one of the following conditions holds.*

(i) *There exists a constant $C > 0$ such that*

$$\nu(E) \le C\,\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(E) \quad \text{for every Borel set } E \subset \partial\Omega.$$

Here $(p+q)'$ denotes the conjugate exponent of $p + q$.

(ii) *There exists a positive constant $C > 0$ such that*

$$\mathbb{N}_{2\alpha-1,1}[\delta^{\alpha p+(\alpha-1)q+\alpha}\mathbb{N}_{2\alpha-1,1}[\nu]^{p+q}] \le C\mathbb{N}_{2\alpha-1,1}[\nu] < \infty \quad \text{a.e. in } \Omega. \tag{1.22}$$

Then there exists $\varrho_0 = \varrho_0(N, \mu, p, q, C, \Omega) > 0$ such that, for any $\varrho \in (0, \varrho_0)$, problem $(P_-^{\varrho\nu})$ admits a weak solution u satisfying

$$|u| \le C'\delta^\alpha\mathbb{N}_{2\alpha-1,1}[\varrho\nu], \quad |\nabla u| \le C'\delta^{\alpha-1}\mathbb{N}_{2\alpha-1,1}[\varrho\nu] \quad \text{in } \Omega, \tag{1.23}$$

where $C' = C'(N, \mu, \Omega)$ is a positive constant.

Organization of the paper. In Section 2, we recall main properties of the first eigenvalue and the corresponding eigenfunction of $-L_\mu$ in Ω and collect estimates on the Green kernel and the Martin kernel, as well as their gradient. In Section 3, we prove Theorem 1.3, and in Section 4, we demonstrate Theorem 1.5 and Theorem 1.6. In Section 5, we give the proof of Theorem 1.7 and Theorem 1.8. Section 6 is devoted to the proof of Theorem 1.9, and in Section 7, the proof of Theorem 1.11 is provided. In Appendix A, we construct a barrier in the case $g(u, |\nabla u|) = |u|^p|\nabla u|^q$. Finally, in Appendix B, we discuss the case $g(u, |\nabla u|) = |u|^p + |\nabla u|^q$ and state main results without proofs since the arguments are similar to those in the case $g(u, |\nabla u|) = u^p|\nabla u|^q$.

1.3 Notations

We list below some notations that we use frequently in the present paper.

- For $\phi \ge 0$, denote by $L^\kappa(\Omega, \phi)$ $(\kappa > 1)$ the space of functions v satisfying $\int_\Omega |v|^\kappa\phi\,dx < \infty$. We denote by $H^1(\Omega, \phi)$ the space of functions v such that $v \in L^2(\Omega, \phi)$ and $\nabla v \in L^2(\Omega, \phi)$. Let $\mathfrak{M}(\Omega, \phi)$ be the space of Radon measures τ on Ω satisfying $\int_\Omega \phi\,d|\tau| < \infty$, and let $\mathfrak{M}^+(\Omega, \phi)$ be the positive cone of $\mathfrak{M}(\Omega, \phi)$. Denote by $\mathfrak{M}(\partial\Omega)$ the space of bounded Radon measures on $\partial\Omega$ and by $\mathfrak{M}^+(\partial\Omega)$ the positive cone of $\mathfrak{M}(\partial\Omega)$.

- Denote $L_w^\kappa(\Omega, \tau)$, $1 \le \kappa < \infty$, $\tau \in \mathfrak{M}^+(\Omega)$, the weak L^κ space (or Marcinkiewicz space) with weight τ. The subscript w is an abbreviation of "weak". See Subsection 2.2 for more details.
- We denote by λ_μ the first eigenvalue of $-L_\mu$ and by φ_μ the corresponding eigenfunction (see Subsection 1.4).
- For $\kappa > 1$, we denote by κ' the conjugate exponent.
- Throughout the paper, c, c_1, c_2, C, C_1, C' denote positive constants which may vary from line to line. We write $C = C(a, b)$ to emphasize the dependence of C on the data a, b.
- The notation $f \approx h$ means that there exist positive constants c_1, c_2 such that $c_1 h < f < c_2 h$.
- Denote by χ_E the indicator function of a set E.
- For $z \in \partial\Omega$, denote by $\mathbf{n}_z$ the outer unit normal vector at z.

2 Preliminaries

We recall that, throughout the paper, we assume that $\mu \in (0, \frac{1}{4}]$ and $\lambda_\mu > 0$.

2.1 Eigenvalue and Eigenfunction

We recall important facts of the eigenvalue λ_μ of $-L_\mu$ and the associated eigenfunction φ_μ which can be found in [13, 14]. If $0 < \mu < \frac{1}{4}$, then the minimizer $\varphi_\mu \in H_0^1(\Omega)$ of (1.4) exists and satisfies $\varphi_\mu \approx \delta^\alpha$, where α is defined in (1.7). If $\mu = \frac{1}{4}$, there is no minimizer of (1.4) in $H_0^1(\Omega)$, but there exists a nonnegative function $\varphi_{\frac{1}{4}} \in H^1_{\mathrm{loc}}(\Omega)$ such that $\varphi_{\frac{1}{4}} \approx \delta^{\frac{1}{2}}$ and it satisfies $-L_{\frac{1}{4}}\varphi_{\frac{1}{4}} = \lambda_\mu \varphi_{\frac{1}{4}}$ in Ω in the sense of distributions. In addition, we have $\delta^{-\frac{1}{2}}\varphi_{\frac{1}{4}} \in H_0^1(\Omega, \delta)$.

2.2 Green Kernel and Martin Kernel

Denote by G_μ^Ω and K_μ^Ω the Green kernel and the Martin kernel of $-L_\mu$ in Ω respectively (see [17, 25]). The Green operator and the Martin operator are defined as follows:

$$\mathbb{G}_\mu^\Omega[\tau](x) := \int_\Omega G_\mu^\Omega(x, y)\, d\tau(y) \quad \text{for every } \tau \in \mathfrak{M}(\Omega, \delta^\alpha),$$

$$\mathbb{K}_\mu^\Omega[\nu](x) := \int_{\partial\Omega} K_\mu^\Omega(x, z)\, d\nu(z) \quad \text{for every } \nu \in \mathfrak{M}(\partial\Omega).$$

When there is no ambiguity, we will drop the superscript Ω, i.e. we write G_μ, K_μ, $\mathbb{G}_\mu$, $\mathbb{K}_\mu$ instead of G_μ^Ω, K_μ^Ω, $\mathbb{G}_\mu^\Omega$, $\mathbb{K}_\mu^\Omega$.

By [14, Theorem 4.11], it holds

$$G_\mu(x, y) \approx \min\{|x-y|^{2-N}, \delta(x)^\alpha \delta(y)^\alpha |x-y|^{2-N-2\alpha}\} \quad \text{for every } x, y \in \Omega,\ x \ne y. \tag{2.1}$$

Since (1.5) holds, by [3] and [17, Proposition 2.29], the Martin kernel K_μ exists. Moreover, it holds (see [25, (2.7), page 76] for $\mu < C_H(\Omega)$ and [17, Theorem 2.30] for $\mu = \frac{1}{4}$)

$$K_\mu(x, y) \approx \delta(x)^\alpha |x-y|^{2-N-2\alpha} \quad \text{for every } x \in \Omega,\ y \in \partial\Omega. \tag{2.2}$$

For estimates on the Green kernel and the Martin kernel of a more general Schrödinger operator, we refer to [21].

Next we recall estimates of Green kernel and Martin kernel in weak L^κ spaces. Let $\tau \in \mathfrak{M}^+(\Omega)$. For $\kappa > 1$, $\kappa' = \frac{\kappa}{\kappa-1}$ and $u \in L^1_{\mathrm{loc}}(\Omega, \tau)$, we set

$$\|u\|_{L_w^\kappa(\Omega,\tau)} := \inf\left\{c \in [0,\infty] : \int_E |u|\, d\tau \le c\left(\int_E d\tau\right)^{\frac{1}{\kappa'}} \text{ for any Borel set } E \subset \Omega\right\}$$

and

$$L_w^\kappa(\Omega,\tau) := \{u \in L^1_{\mathrm{loc}}(\Omega,\tau) : \|u\|_{L_w^\kappa(\Omega,\tau)} < \infty\}.$$

$L_w^\kappa(\Omega,\tau)$ is called weak L^κ space (or Marcinkiewicz space with exponent κ) with quasi-norm $\|\cdot\|_{L_w^\kappa(\Omega,\tau)}$. See [29] for more details. Notice that, for every $s > -1$,

$$L_w^\kappa(\Omega,\delta^s) \subset L^r(\Omega,\delta^s) \quad \text{for every } r \in [1,\kappa). \tag{2.3}$$

Moreover, for any $u \in L_w^\kappa(\Omega,\delta^s)$ $(s > -1)$,

$$\int_{\{|u|\geq\lambda\}} \delta^s\,dx \leq \lambda^{-\kappa}\|u\|^\kappa_{L_w^\kappa(\Omega,\delta^s)} \quad \text{for all } \lambda > 0. \tag{2.4}$$

Proposition 2.1 ([15, Proposition 2.4]). *The following statements hold.*
(i) *Let* $\gamma \in (-\frac{\alpha N}{N+2\alpha-2}, \frac{\alpha N}{N-2})$. *There exists a constant* $c = c(N,\mu,\gamma,\Omega)$ *such that*

$$\|\mathbb{G}_\mu[\tau]\|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega,\delta^\gamma)} \leq c\|\tau\|_{\mathfrak{M}(\Omega,\delta^\alpha)} \quad \textit{for all } \tau \in \mathfrak{M}(\Omega,\delta^\alpha). \tag{2.5}$$

(ii) *Let* $\gamma > -1$. *Then there exists a constant* $c = c(N,\mu,\gamma,\Omega)$ *such that*

$$\|\mathbb{K}_\mu[\nu]\|_{L_w^{\frac{N+\gamma}{N+\alpha-2}}(\Omega,\delta^\gamma)} \leq c\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \textit{for all } \nu \in \mathfrak{M}(\partial\Omega). \tag{2.6}$$

Proposition 2.2 ([16, Proposition A]). *The following statements hold.*
(i) *Let* $\theta \in [0,\alpha]$ *and* $\gamma \in [0, \frac{\theta N}{N-1})$. *Then there exists a positive constant* $c = c(N,\mu,\theta,\gamma,\Omega)$ *such that*

$$\|\nabla\mathbb{G}_\mu[|\tau|]\|_{L_w^{\frac{N+\gamma}{N+\theta-1}}(\Omega,\delta^\gamma)} \leq c\|\tau\|_{\mathfrak{M}(\Omega,\delta^\theta)} \quad \textit{for all } \tau \in \mathfrak{M}(\Omega,\delta^\theta), \tag{2.7}$$

where $\nabla\mathbb{G}_\mu[\tau](x) = \int_\Omega \nabla_x G_\mu(x,y)\,d\tau(y)$.
(ii) *Let* $\gamma \geq 0$. *Then there exists a positive constant* $c = c(N,\mu,\gamma,\Omega)$ *such that*

$$\|\nabla\mathbb{K}_\mu[|\nu|]\|_{L_w^{\frac{N+\gamma}{N+\alpha-1}}(\Omega,\delta^\gamma)} \leq c\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \textit{for all } \nu \in \mathfrak{M}(\partial\Omega),$$

where $\nabla\mathbb{K}_\mu[\nu](x) = \int_{\partial\Omega} \nabla_x K_\mu(x,z)\,d\nu(z)$.

2.3 Linear Equations

The Green kernel and the Martin kernel play an important role in the study of the boundary value problem for the linear equation

$$\begin{cases} -L_\mu u = \tau & \text{in } \Omega, \\ \operatorname{tr}(u) = \nu. \end{cases} \tag{2.8}$$

Definition 2.3. Assume $(\tau,\nu) \in \mathfrak{M}(\Omega,\delta^\alpha) \times \mathfrak{M}(\partial\Omega)$. We say that u is a *weak solution* of (2.8) if $u \in L^1(\Omega,\delta^\alpha)$ and

$$-\int_\Omega uL_\mu\zeta\,dx = \int_\Omega \zeta\,d\tau - \int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta\,dx \quad \text{for all } \zeta \in \mathbf{X}_\mu(\Omega),$$

where $\mathbf{X}_\mu(\Omega)$ is defined in (1.9).

Proposition 2.4 ([15, Proposition 2.11]). *Assume that* $(\tau,\nu) \in \mathfrak{M}(\Omega,\delta^\alpha) \times \mathfrak{M}(\partial\Omega)$. *Then u is a weak solution of* (2.8) *if and only if* $u = \mathbb{G}_\mu[\tau] + \mathbb{K}_\mu[\nu]$ *in* Ω. *Moreover, there exists a constant* $C = C(N,\mu,\Omega) > 0$ *such that* $\|u\|_{L^1(\Omega,\delta^\alpha)} \leq C(\|\tau\|_{\mathfrak{M}(\Omega,\delta^\alpha)} + \|\nu\|_{\mathfrak{M}(\partial\Omega)})$.

For $D \Subset \Omega$, denote by G_μ^D and K_μ^D the Green kernel and the Poisson kernel of $-L_\mu$ in D. Consider the problem

$$\begin{cases} -L_\mu u = \varphi & \text{in } D, \\ \quad\ \ u = \eta & \text{on } \partial D. \end{cases} \tag{2.9}$$

A similar result has been established for (2.9).

Proposition 2.5 ([17]). *For any $(\varphi, \eta) \in \mathfrak{M}(D, \delta_D) \times \mathfrak{M}(\partial D)$ (where $\delta_D = \operatorname{dist}(\cdot, D)$), there exists a unique weak solution $u = u_{\varphi,\eta} \in L^1(D, \delta_D)$ of (2.9), i.e.*

$$-\int_D uL_\mu \zeta \, dx = \int_D \zeta \, d\varphi - \int_D \mathbb{K}_\mu^D[\eta] L_\mu \zeta \, dx \quad \text{for all } \zeta \in \mathbf{X}_0(D),$$

where $\mathbf{X}_0(D)$ is defined similarly to $\mathbf{X}_\mu(\Omega)$ with $\mu = 0$ and $\Omega = D$. It holds

$$u = \mathbb{G}_\mu^D[\varphi] + \mathbb{K}_\mu^D[\eta], \tag{2.10}$$

and there exists a constant $c = c(N, \mu, D) > 0$ such that $\|u\|_{L^1(D,\delta_D)} \leq c(\|\varphi\|_{\mathfrak{M}(\Omega,\delta_D)} + \|\eta\|_{\mathfrak{M}(\partial D)})$.

Finally, we will need the following classical properties of C^2 domains.

Proposition 2.6 ([29]). *There exists a positive constant β_0 such that $\delta \in C^2(\overline{\Omega}_{4\beta_0})$. Moreover, for any $x \in \Omega_{4\beta_0}$, there exists a unique $\xi_x \in \partial\Omega$ such that*
(a) *$\delta(x) = |x - \xi_x|$ and $\mathbf{n}_{\xi_x} = -\nabla\delta(x) = -\frac{x-\xi_x}{|x-\xi_x|}$, where $\mathbf{n}_{\xi_x}$ denotes the outer unit normal vector at $\xi_x \in \partial\Omega$,*
(b) *$x(s) := x + s\nabla\delta(x) \in \Omega_{\beta_0}$ and $\delta(x(s)) = |x(s) - \xi_x| = \delta(x) + s$ for any $0 < s < 4\beta_0 - \delta(x)$.*

3 Nonlinear Equations with Subcritical Absorption

In this section, we establish the existence of a positive solution of (P_+^ν). The approach is based on a combination of the idea in [20], estimates on the Green kernel, the Martin kernel, their gradient and the Vitali convergence theorem.

Proof of Theorem 1.3. We divide the proof into three steps.

Step 1. In this step, we assume that

$$M := \sup_{s\in\mathbb{R}, t\in\mathbb{R}_+} |g(s,t)| < +\infty. \tag{3.1}$$

Let $D \subset\subset \Omega$ be a smooth open domain, and consider the equation

$$-L_\mu v + g(v + \mathbb{K}_\mu[v], |\nabla(v + \mathbb{K}_\mu[v])|) = 0 \quad \text{in } D. \tag{3.2}$$

First we note that $u_1 = 0$ is a supersolution of (3.2) and $u_2 = -\mathbb{K}_\mu[v]$ is a solution of (3.2). Let

$$\mathcal{T}(u) := \begin{cases} 0 & \text{if } 0 \leq u, \\ u & \text{if } u_2 \leq u \leq 0, \\ u_2 & \text{if } u \leq u_2. \end{cases} \tag{3.3}$$

In this step, we use the idea in [20] in order to construct a solution $v \in W^{1,\infty}(D)$ of the problem

$$\begin{cases} -L_\mu v + g(v + \mathbb{K}_\mu[v], |\nabla(v + \mathbb{K}_\mu[v])|) = 0 & \text{in } D, \\ v = 0 & \text{on } \partial D, \end{cases} \tag{3.4}$$

which satisfies

$$-\mathbb{K}_\mu[v] \leq v \leq 0 \quad \text{for all } x \in D. \tag{3.5}$$

Let $d_{D,\Omega} := \operatorname{dist}(\partial D, \partial\Omega)$ and $u \in W^{1,1}(D)$. By the standard elliptic theory, there exists a unique solution of the problem

$$\begin{cases} -\Delta w + (d_{D,\Omega}^{-2} - \mu\delta^{-2})w = -g(v + \mathbb{K}_\mu[v], |\nabla(v + \mathbb{K}_\mu[v])|) + d_{D,\Omega}^{-2}\mathcal{T}(v) & \text{in } D, \\ w = 0 & \text{on } \partial D. \end{cases} \tag{3.6}$$

Recall that $\delta = \operatorname{dist}(\cdot, \partial\Omega)$.

We define an operator $\mathbb{A}$ as follows: to each $u \in W^{1,1}(D)$, we associate the unique solution $\mathbb{A}[u]$ of (3.6). Furthermore, since

$$d_{D,\Omega}^{-2} - \mu\delta(x)^{-2} \geq (1-\mu)\delta(x)^{-2} \quad \text{for all } x \in D,$$

by standard elliptic estimates, there exists a constant $C_1 = C_1(N, \mu, d_{D,\Omega}, D) > 0$ such that

$$\sup_{x\in D}|\mathbb{A}[u](x)| \leq C_1(M + \|\nu\|_{\mathfrak{M}(\partial\Omega)}) =: A_1.$$

Also, by (3.3) and standard elliptic estimates, there exists a positive constant $C_2 = C_2(N, \mu, d_{D,\Omega}, D)$ such that

$$\sup_{x\in D}|\nabla\mathbb{A}[u](x)| \leq C_2(M + \|\nu\|_{\mathfrak{M}(\partial\Omega)}) =: A_2.$$

By using an argument similar to the proof of [16, Theorem B, Step 1], we can show that

$$\mathbb{A}: W^{1,1}(D) \to W^{1,1}(D)$$

is continuous and compact. Now set $\mathcal{K} := \{\xi \in W^{1,1}(D) : \|\xi\|_{W^{1,\infty}(D)} \leq A_1 + A_2\}$. Then $\mathcal{K}$ is a closed, convex subset of $W^{1,1}(D)$ and $\mathbb{A}(\mathcal{K}) \subset \mathcal{K}$. Thus we can apply the Schauder fixed point theorem to obtain the existence of a function $v \in \mathcal{K}$ such that $\mathbb{A}[v] = v$. This means v is a weak solution of (3.6). By the standard elliptic theory, we can easily deduce that $v, u_2 \in C^2(D) \cap C(\overline{D})$. Moreover, it can be seen that $v \leq 0$.

Now we allege that $v \geq u_2$ in D by employing an argument of contradiction. Suppose by contradiction that there exists $x_0 \in D$ such that $\inf_{x\in D}(v(x) - u_2(x)) = v(x_0) - u_2(x_0) < 0$. Then we have $\nabla v(x_0) = \nabla u_2(x_0)$, $-\Delta(v - u_2)(x_0) \leq 0$, $\mathcal{T}[v](x_0) = \mathcal{T}[u_2](x_0) = u_2(x_0)$. But

$$\begin{aligned}-\Delta(v-u_2)(x_0) = &-(d_{D,\Omega}^{-2} - \mu\delta(x_0)^{-2})(v(x_0) - u_2(x_0))\\ &- g(v(x_0) + \mathbb{K}_\mu[\nu](x_0), |\nabla v(x_0) + \nabla\mathbb{K}_\mu[\nu](x_0)|)\\ &+ g(u_2(x_0) + \mathbb{K}_\mu[\nu](x_0), |\nabla u_2(x_0) + \nabla\mathbb{K}_\mu[\nu](x_0)|) > 0,\end{aligned}$$

which is clearly a contradiction. Therefore, $v \geq u_2$ in D.

As a consequence, $\mathcal{T}(v) = v$, and therefore v is a solution of (3.4).

Step 2. In this step, we still assume that (3.1) holds. Let $\{\Omega_n\}$ be a smooth exhaustion of Ω, and let v_n be the solution of (3.4) in $D = \Omega_n$ (constructed in Step 1) satisfying (3.5). Then there exists a constant $C = C(N, \mu, \Omega) > 0$ such that

$$|v_n(x)| \leq \mathbb{G}_\mu\big[\chi_{\Omega_n} g\big(v_n + \mathbb{K}_\mu[\nu], |\nabla(v_n + \mathbb{K}_\mu[\nu])|\big)\big](x) \leq CM\delta(x)^\alpha \quad \text{for all } x \in \Omega_n.$$

This implies that there exists a subsequence, still denoted by $\{v_n\}$, such that $v_n \to v$ in $W^{1,p}_{\mathrm{loc}}(\Omega)$ and v satisfies

$$\begin{cases} -L_\mu v + g\left(v + \mathbb{K}_\mu[\nu], |\nabla(v + \mathbb{K}_\mu[\nu])|\right) = 0 & \text{in } \Omega,\\ \mathrm{tr}(v) = 0. \end{cases}$$

Furthermore, $-\mathbb{K}_\mu[\nu] \leq v \leq 0$ for all $x \in \Omega$. Setting $u = v + \mathbb{K}_\mu[\nu]$, then u is a solution of (P_+^ν) satisfying $0 \leq u \leq \mathbb{K}_\mu[\nu]$ in Ω.

Step 3. In this step, we drop condition (3.1). Set $g_n := \min(g, n)$, and let u_n be a nonnegative solution (the existence of u_n is guaranteed in Step 2) of

$$\begin{cases} -L_\mu u_n + g_n(u_n, |\nabla u_n|) = 0 & \text{in } \Omega,\\ \mathrm{tr}(u_n) = \nu \end{cases}$$

satisfying

$$0 \leq u_n \leq \mathbb{K}_\mu[\nu] \quad \text{in } \Omega. \tag{3.7}$$

Then u_n satisfies

$$-\int_\Omega u_n L_\mu\zeta\,dx + \int_\Omega g_n(u_n, |\nabla u_n|)\zeta\,dx = -\int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta\,dx \quad \text{for all } \zeta \in \mathbf{X}_\mu(\Omega), \tag{3.8}$$

$$u_n + \mathbb{G}_\mu[g_n(u_n, |\nabla u_n|)] = \mathbb{K}_\mu[\nu]. \tag{3.9}$$

Choosing $\zeta = \varphi_\mu$, where φ_μ is an eigenfunction associated to the first eigenvalue of $-L_\mu$, by (3.8), we have

$$\lambda_\mu \int_\Omega |u_n|\varphi_\mu \, dx + \int_\Omega g_n(u_n, |\nabla u_n|)\varphi_\mu \, dx \le \lambda_\mu \int_\Omega \mathbb{K}_\mu[|\nu|]\varphi_\mu \, dx. \tag{3.10}$$

Now, by (3.9) and Proposition 2.2, we obtain

$$\|\nabla u_n\|_{L_w^{q_\mu}(\Omega,\delta^\alpha)} \le c(N,\mu,\Omega)\big(\|g_n(u_n,|\nabla u_n|)\|_{L^1(\Omega,\delta^\alpha)} + \|\nu\|_{\mathfrak{M}(\partial\Omega)}\big).$$

This, together with (3.10) and (2.6), implies

$$\|\nabla u_n\|_{L_w^{q_\mu}(\Omega,\delta^\alpha)} \le c(N,\mu,\Omega)\|\nu\|_{\mathfrak{M}(\partial\Omega)}.$$

Similarly, we can show that

$$\|u_n\|_{L_w^{p_\mu}(\Omega,\delta^\alpha)} \le c(N,\mu,\Omega)\|\nu\|_{\mathfrak{M}(\partial\Omega)}.$$

Next we prove that

$$g_n(u_n,|\nabla u_n|) \to g(u,|\nabla u|) \quad \text{in } L^1(\Omega,\delta^\alpha). \tag{3.11}$$

For $\lambda > 0$ and any function w, set

$$\begin{aligned}
&\mathbf{A}_\lambda^w := \{x \in \Omega : |w(x)| > \lambda\}, && \mathbf{a}^w(\lambda) := \int_{\mathbf{A}_\lambda^w} \delta^\alpha \, dx,\\
&\mathbf{B}_\lambda^w := \{x \in \Omega : |\nabla w(x)| > \lambda^{\frac{p_\mu}{q_\mu}}\}, && \mathbf{b}^w(\lambda) := \int_{\mathbf{B}_\lambda^w} \delta^\alpha \, dx,\\
&\mathbf{C}_\lambda^w := \mathbf{A}_\lambda^w \cap \mathbf{B}_\lambda^w, && \mathbf{c}^w(\lambda) := \int_{\mathbf{C}_\lambda^w} \delta^\alpha \, dx.
\end{aligned} \tag{3.12}$$

Then, for $\lambda > 0$ and $n \in \mathbb{N}$, put

$$\begin{aligned}
\mathbf{A}_{n,\lambda} &= \mathbf{A}_\lambda^{u_n}, \quad \mathbf{a}_n(\lambda) = \mathbf{a}^{u_n}(\lambda),\\
\mathbf{B}_{n,\lambda} &= \mathbf{B}_\lambda^{u_n}, \quad \mathbf{b}_n(\lambda) = \mathbf{b}^{u_n}(\lambda),\\
\mathbf{C}_{n,\lambda} &= \mathbf{C}_\lambda^{u_n}, \quad \mathbf{c}_n(\lambda) = \mathbf{c}^{u_n}(\lambda).
\end{aligned}$$

For any Borel set $E \subset \Omega$,

$$\begin{aligned}
\int_E g_n(u_n,|\nabla u_n|)\delta^\alpha \, dx &= \int_{E\cap\mathbf{C}_{n,\lambda}} g_n(u_n,|\nabla u_n|)\delta^\alpha \, dx + \int_{E\cap\mathbf{A}_{n,\lambda}^c\cap\mathbf{B}_{n,\lambda}} g_n(u_n,|\nabla u_n|)\delta^\alpha \, dx\\
&\quad + \int_{E\cap\mathbf{A}_{n,\lambda}\cap\mathbf{B}_{n,\lambda}^c} g_n(u_n,|\nabla u_n|)\delta^\alpha \, dx + \int_{E\cap\mathbf{A}_{n,\lambda}^c\cap\mathbf{B}_{n,\lambda}^c} g_n(u_n,|\nabla u_n|)\delta^\alpha \, dx\\
&\le C\int_\lambda^\infty g(s, s^{\frac{p_\mu}{q_\mu}})s^{-1-p_\mu} \, ds + g(\lambda, \lambda^{\frac{p_\mu}{q_\mu}})\int_E \delta^\alpha \, dx.
\end{aligned} \tag{3.13}$$

Note that the first term on the right-hand side of (3.13) tends to 0 as $\lambda \to \infty$. Therefore, for any $\varepsilon > 0$, there exists $\lambda > 0$ such that the first term on the right-hand side of (3.13) is smaller than $\frac{\varepsilon}{2}$. Fix such λ, and put

$$\eta = \frac{\varepsilon}{2\max\{g(\lambda, \lambda^{\frac{p_\mu}{q_\mu}}), 1\}}.$$

Then, by (3.13),

$$\int_E \delta^\alpha \, dx \le \eta \implies \int_E g_n(u_n,|\nabla u_n|)\delta^\alpha \, dx < \varepsilon.$$

Therefore, the sequence $\{g_n(u_n,|\nabla u_n|)\}$ is equi-integrable in $L^1(\Omega,\delta^\alpha)$. Thus, by invoking the Vitali convergence theorem, we derive (3.11).

From (3.7), we deduce that $0 \le u_n \to u$ in $L^1(\Omega,\delta^\alpha)$. Therefore, letting $n \to \infty$ in (3.8), we deduce that u is a weak solution of (P_+^ν). □

4 Absorption $g(u, |\nabla u|) = |u|^p|\nabla u|^q$: Subcritical Case

In this section, we assume $g(u, |\nabla u|) = |u|^p|\nabla u|^q$ with $p \ge 0$, $q > 0$, $p + q > 1$. We recall that (see Remark 1.4) g satisfies (1.10) if and only if (1.11) holds. Moreover, g satisfies (1.21). Therefore, by Theorem 1.3, for any $\nu \in \mathfrak{M}^+(\partial\Omega)$, the problem

$$\begin{cases} -L_\mu u + |u|^p|\nabla u|^q = 0 & \text{in } \Omega, \\ \qquad\qquad \operatorname{tr}(\nu) = \nu \end{cases} \tag{4.1}$$

admits a positive weak solution.

Next we prove the following regularity result.

Proposition 4.1. *Assume $p \ge 0$ and $0 < q < \frac{N}{N-1}$. If u is a nonnegative solution of*

$$-L_\mu u + |u|^p|\nabla u|^q = 0 \quad \text{in } \Omega, \tag{4.2}$$

then $u \in C^2(\Omega)$.

Proof. Let $D \subset\subset \Omega$ be a smooth open domain. Since u is a nonnegative solution of (4.1), by (2.10), we can easily obtain $0 \le u(x) \le \mathbb{K}_\mu[\nu](x) \le C_D$ for all $x \in D$. Consequently, $|u|^p|\nabla u|^q \le C_D^p|\nabla u|^q$ in D. Hence, by invoking [16, Lemma 4.2], we can derive the desired result. □

4.1 Comparison Principle

Lemma 4.2. *Let $u \in C^2(\Omega)$ be a nonnegative solution of (4.2). If there exists $x_0 \in \Omega$ such that $u(x_0) = 0$, then $u \equiv 0$.*

Proof. By Young's inequality, $|u|^p|\nabla u|^q \le |u|^{p+q} + |\nabla u|^{p+q}$ in Ω. As a consequence, u satisfies

$$-L_\mu u + |u|^{p+q} + |\nabla u|^{p+q} \ge 0 \quad \text{in } \Omega. \tag{4.3}$$

Now set $\mathbf{a}(x) := |\nabla u(x)|^{p+q-2}\nabla u(x)$ and $b(x) := |u(x)|^{p+q-1}$. Let $\beta \in (0, \beta_0)$ be small enough such that $x_0 \in D_\beta$. Since $u \in C^2(\Omega)$, there exists a constant C_β such that $\sup_{x\in D_\beta}|\mathbf{a}(x)| + \sup_{x\in D_\beta} b(x) \le C_\beta$. From (4.3), we deduce $-\Delta u + \mathbf{a}\cdot\nabla u + bu \ge \frac{\mu}{\delta^2}u \ge 0$ in D_β. Since $b(x) \ge 0$, by the maximum principle, u cannot achieve a nonpositive minimum in D_β. Thus the result follows straightforward. □

Next we state the comparison principle for (4.2).

Lemma 4.3. *Let $p \ge 0$, $q \ge 1$ and $D \subset \Omega$. We assume that $u_1, u_2 \in C^2(D)$ are respectively nonnegative subsolution and positive supersolution of (4.2) in D such that*

$$\limsup_{x\to\partial D} \frac{u_1(x)}{u_2(x)} < 1. \tag{4.4}$$

Then $u_1 \le u_2$ in D.

Proof. Suppose by contradiction that

$$m := \sup_{x\in D} \frac{u_1(x)}{u_2(x)} > 1.$$

By (4.4), we deduce that there exists $x_0 \in D$ such that

$$\frac{u_1(x_0)}{u_2(x_0)} = \sup_{x\in D}\frac{u_1(x)}{u_2(x)} = m.$$

Let $r > 0$ be such that $B(x_0, r) \subset D$. Then we see that

$$-\Delta(m^{-1}u_1 - u_2) + (m^{-1}u_1)^p|m^{-1}\nabla u_1|^q - u_2^p|\nabla u_2|^q \le \frac{\mu}{\delta^2}(m^{-1}u_1 - u_2) \le 0 \quad \text{in } B\Big(x_0, \frac{r}{2}\Big). \tag{4.5}$$

Now note that

$$\begin{aligned}(m^{-1}u_1)^p|m^{-1}\nabla u_1|^q - u_2^p|\nabla u_2|^q &= (m^{-1}u_1)^p|m^{-1}\nabla u_1|^q - (m^{-1}u_1)^p|\nabla u_2|^q \\ &\quad + (m^{-1}u_1)^p|\nabla u_2|^q - u_2^p|\nabla u_2|^q \\ &= \tilde{\mathbf{a}}(x)(m^{-1}\nabla u_1 - \nabla u_2) + \tilde{b}(x)(m^{-1}u_1 - u_2),\end{aligned} \tag{4.6}$$

where

$$\tilde{\mathbf{a}}(x) = (m^{-1}u_1)^p\frac{|m^{-1}\nabla u_1|^q - |\nabla u_2|^q}{|m^{-1}\nabla u_1 - \nabla u_2|^2}(m^{-1}\nabla u_1 - \nabla u_2)$$

and

$$\tilde{b}(x) := |\nabla u_2|^q\Big(\frac{(m^{-1}u_1)^p - u_2^p}{m^{-1}u_1 - u_2}\Big) \geq 0.$$

Since $u_1, u_2 \in C^2(D)$, $u_2(x) > 0$ for any $x \in D$ and $q \geq 1$, there exists a positive constant $C > 0$ such that

$$\sup_{x\in B(x_0,\frac{r}{2})}|\tilde{\mathbf{a}}(x)| + \sup_{x\in B(x_0,\frac{r}{2})}\tilde{b}(x) < C.$$

Combining (4.5) and (4.6), we have

$$-\Delta(m^{-1}u_1 - u_2) + \tilde{\mathbf{a}}\cdot\nabla(m^{-1}u_1 - u_2) + \tilde{b}(m^{-1}u_1 - u_2) \leq 0 \quad \text{in } B\Big(x_0, \frac{r}{2}\Big).$$

Hence, by the maximum principle, $m^{-1}u_1 - u_2$ cannot achieve a nonnegative maximum in $B(x_0, \frac{r}{2})$. This is a contradiction. Thus $u_1 \leq u_2$ in D. □

In order to prove the comparison principle for (4.1), we need the following result.

Lemma 4.4. [16, Lemma 4.5] *Let $\tau \in \mathfrak{M}(\Omega, \delta^\alpha)$, and $v \geq 0$ satisfies*

$$\begin{cases}-L_\mu v \leq \tau & \text{in } \Omega,\\ \operatorname{tr}(v) = 0.\end{cases}$$

Then, for any $1 < \kappa < q_\mu$, there exists a constant $c = c(N, \Omega, \mu)$ such that $\|\nabla v\|_{L^\kappa(\Omega,\delta^\alpha)} \leq c\|\tau\|_{\mathfrak{M}(\Omega,\delta^\alpha)}$.

Proof of Theorem 1.5. Since u_i is a nonnegative solution of (4.1), $|u_i|^p|\nabla u_i|^q \in L^1(\Omega, \delta^\alpha)$, $i = 1, 2$. Moreover, from Propositions 2.1 and 2.2, we deduce that

$$\|u_i\|_{L^{p_1}(\Omega,\delta^\alpha)} + \|\nabla u_i\|_{L^{q_1}(\Omega,\delta^\alpha)} \leq c_1(\||u_i|^p|\nabla u_i|^q\|_{L^1(\Omega,\delta^\alpha)} + \|\nu_i\|_{\mathfrak{M}(\partial\Omega)})$$

for any $1 < p_1 < p_\mu$, $1 < q_1 < q_\mu$ and $i = 1, 2$.

Without loss of generality, we assume that $\nu_2 \neq 0$; thus, by Lemma 4.2, $u_2 > 0$ in Ω. In addition, by Proposition 4.1, $u_i \in C^2(\Omega)$ for $i = 1, 2$. Finally, by the representation formula, we have

$$u_i + \mathbb{G}_\mu[|u_i|^p|\nabla u_i|^q] = \mathbb{K}_\mu[\nu_i], \quad i = 1, 2.$$

Let $0 < \varepsilon \leq 1$. Then

$$(\varepsilon u_1 - u_2)_+ \leq \big(\mathbb{G}_\mu[|u_2|^p\,\nabla u_2|^q] - \varepsilon\mathbb{G}_\mu[|u_1|^p|\nabla u_1|^q]\big)_+ \leq \mathbb{G}_\mu\big[\big||u_2|^p|\nabla u_2|^q - |\varepsilon u_1|^p|\nabla u_1|^q\big|\big] =: v,$$

which implies $\operatorname{tr}((\varepsilon u_1 - u_2)_+) \leq \operatorname{tr}(v) = 0$. Hence $\operatorname{tr}((\varepsilon u_1 - u_2)_+) = 0$.

Note that εu_1 is a subsolution of (4.2). Also, since $u_i \in C^2(\Omega)$ and $u_2 > 0$ in Ω, it follows that, for small enough $\beta > 0$,

$$C_\beta := \sup_{x\in D_\beta}\frac{u_1}{u_2} < \infty.$$

Without loss of generality, we assume that $C_\beta > 1$. Set $\varepsilon_\beta = \frac{1}{C_\beta} < 1$. Then $\varepsilon_\beta u_1 - u_2 \leq 0$ in D_β. Moreover, in view of the proof of Lemma 4.3, we derive that

$$\varepsilon_\beta u_1 - u_2 < 0 \quad \text{in } D_\beta. \tag{4.7}$$

Put $E_\beta := \{x \in \Omega : \varepsilon_\beta u_1 - u_2 > 0\}$. Due to Kato's inequality [29], we get

$$-L_\mu(\varepsilon_\beta u_1 - u_2)_+ \le (u_2^p|\nabla u_2|^q - (\varepsilon u_1)^p|\varepsilon\nabla u_1|^q)\chi_{E_\beta} \le \big((\varepsilon u_1)^p(|\nabla u_2|^q - |\varepsilon\nabla u_1|^q)\big)\chi_{E_\beta}. \tag{4.8}$$

By (4.7), we derive that $E_\beta \subset \Omega_\beta$.

Let $\kappa > 1$ and

$$\max\Big\{1, \frac{N+\alpha}{N+\alpha-p(N+\alpha-2)}\Big\} < \kappa < \frac{N+\alpha}{q(N+\alpha-1)}.$$

Note that, for this choice of κ, we have

$$\frac{\kappa}{\kappa-1}p < p_\mu \quad \text{and} \quad \kappa q < q_\mu.$$

Using (4.8), Lemma 4.4 and Hölder's inequality, we get

$$\begin{aligned}
\int_\Omega |\nabla(\varepsilon_\beta u_1 - u_2)_+|^{\kappa q}\delta^\alpha\,dx
&\le c\Big(\int_\Omega (\varepsilon_\beta u_1)^p\big||\nabla u_2|^q - |\varepsilon_\beta\nabla u_1|^q\big|\chi_{E_\beta}\delta^\alpha\,dx\Big)^{\kappa q}\\
&\le c\Big(\int_{E_\beta} (\varepsilon_\beta u_1)^p(\varepsilon_\beta^{q-1}|\nabla u_1|^{q-1} + |\nabla u_2|^{q-1})|\nabla(\varepsilon_\beta u_1 - u_2)|\delta^\alpha\,dx\Big)^{\kappa q}\\
&\le c\Big(\int_{E_\beta} (\varepsilon_\beta u_1)^{\frac{\kappa}{\kappa-1}p}\delta^\alpha\,dx\Big)^{q(\kappa-1)}\Big(\int_{E_\beta}\big((\varepsilon_\beta^{q-1}|\nabla u_1|^{q-1} + |\nabla u_2|^{q-1})|\nabla(\varepsilon_\beta u_1 - u_2)|\big)^\kappa\delta^\alpha\,dx\Big)^q\\
&\le c\Big(\int_{E_\beta} (\varepsilon_\beta u_1)^{\frac{\kappa}{\kappa-1}p}\delta^\alpha\,dx\Big)^{q(\kappa-1)}\Big(\int_{E_\beta}(|\nabla u_1|^{\kappa q} + |\nabla u_2|^{\kappa q})\delta^\alpha\,dx\Big)^{q-1}\Big(\int_{E_\beta}(|\nabla(\varepsilon_\beta u_1 - u_2)|)^{\kappa q}\delta^\alpha\,dx\Big).
\end{aligned} \tag{4.9}$$

Since $E_\beta \subset \Omega_\beta$, $u_1 \in L^{\frac{\kappa}{\kappa-1}p}(\Omega, \delta^\alpha)$ and $|\nabla u_i| \in L^{\kappa q}(\Omega, \delta^\alpha)$, we can choose β_* small enough such that

$$c\Big(\int_{E_{\beta_*}} (\varepsilon_{\beta_*} u_1)^{\frac{\kappa}{\kappa-1}p}\delta^\alpha\,dx\Big)^{q(\kappa-1)}\Big(\int_{E_{\beta_*}}(|\nabla u_1|^{\kappa q} + |\nabla u_2|^{\kappa q})\delta^\alpha\,dx\Big)^{q-1} < \frac{1}{4}.$$

By the above inequality and (4.9), we obtain

$$\nabla(\varepsilon_{\beta_*}u_1 - u_2)_+ = 0 \implies (\varepsilon_{\beta_*}u_1 - u_2)_+ = c_*$$

for some constant $c_* \ge 0$, and since $(\varepsilon_{\beta_*}u_1 - u_2)_+ = 0$ on $\overline{D}_{\beta_*}$, we have $c_* = 0$, namely $\varepsilon_{\beta_*}u_1 \le u_2$ in Ω. As a consequence,

$$\sup_{x\in\Omega}\frac{u_1(x)}{u_2(x)} = \sup_{x\in D_{\beta_*}}\frac{u_1(x)}{u_2(x)} = \sup_{x\in\partial D_{\beta_*}}\frac{u_1(x)}{u_2(x)} = \varepsilon_{\beta_*}^{-1} > 1. \tag{4.10}$$

This implies the existence of $x_* \in \partial D_{\beta_*}$ such that

$$(\varepsilon_{\beta_*}u_1 - u_2)(x_*) = 0. \tag{4.11}$$

Next we take $\beta < \beta_*$; then $\varepsilon_\beta \le \varepsilon_{\beta_*}$. On the other hand, we infer from (4.10) that $\varepsilon_\beta \ge \varepsilon_{\beta_*}$ and hence $\varepsilon_\beta = \varepsilon_{\beta_*}$. Therefore, (4.11) contradicts (4.7). The proof is complete. □

Lemma 4.5. *Let $p \ge 0$, $1 < q < \frac{N}{N-1}$ and $p + q > 1$. If u is a nonnegative solution of* (4.2), *then*

$$u(x) \le C\delta(x)^{-\frac{2-q}{q+p-1}} + M_{\beta_0} \quad \text{for all } x \in \Omega, \tag{4.12}$$

$$|\nabla u(x)| \le C'\delta(x)^{-\frac{1+p}{p+q-1}} \quad \text{for all } x \in \Omega, \tag{4.13}$$

where $M_{\beta_0} := \sup_{\overline{D}_{\beta_0}} u$, $C = C(N, \mu, q, p, \beta_0, M_{\beta_0})$ and $C' = C'(N, \mu, q, p, \beta_0, M_{\beta_0})$.

Proof. The proof is similar to that of [16, Lemma 4.6], and hence we omit it. □

4.2 Isolated Singularities

In this section, we assume the origin $0 \in \partial\Omega$ and study the behavior near 0 of solutions of (4.2) which vanish on $\partial\Omega \setminus \{0\}$. We first establish pointwise a priori estimates for solutions with an isolated singularity at 0, as well as their gradient.

Proposition 4.6. *Assume* $0 \in \partial\Omega$, $p \geq 0$, $q \geq 1$, $p + q > 1$ *and* p *and* q *satisfy* (1.11). *Let* u *be a positive solution of* (4.2) *in* Ω *such that*

$$\lim_{x\in\Omega,\, x\to\xi} \frac{u(x)}{W(x)} = 0 \quad \text{for all } \xi \in \partial\Omega \setminus \{0\}, \tag{4.14}$$

locally uniformly in $\partial\Omega \setminus \{0\}$. *Here* W *is defined in* (1.17). *Then there exists a constant* $C = C(N, \mu, q, p, \Omega) > 0$ *such that*

$$u(x) \leq C\delta(x)^{\alpha}|x|^{-\frac{2-q}{p+q-1}-\alpha} \qquad \text{for all } x \in \Omega, \tag{4.15}$$

$$|\nabla u(x)| \leq C\delta(x)^{\alpha-1}|x|^{-\frac{2-q}{p+q-1}-\alpha} \qquad \text{for all } x \in \Omega.$$

Proof. We split the proof into two steps.

Step 1. Let β_0 be the constant in Proposition 2.6. Let $x_i \in \partial\Omega$ be such that $|x_i| \geq \frac{\beta_0}{16}$,

$$\partial\Omega \subset B\Big(0, \frac{\beta_0}{4}\Big) \cup \bigcup_{i=1}^{n} B\Big(x_i, \frac{\beta_0}{32}\Big) =: \mathcal{A} \quad \text{for some } n \in \mathbb{N}.$$

Notice that there exists a constant $\varepsilon_0 = \varepsilon_0(\beta_0) > 0$ such that $\operatorname{dist}(\partial\mathcal{A}, \partial\Omega) > \varepsilon_0$.

Let w_i be the function constructed in Proposition A.1 in $B(x_i, \frac{\beta_0}{16})$ for $R = \frac{\beta_0}{16}$, $i = 1, \dots, n$. Then, by the maximum principle (see [17, Propositions 2.13 and 2.14]), we have

$$u(x) \leq w_i(x) \quad \text{for all } x \in B\Big(x_i, \frac{\beta_0}{16}\Big),\ i = 1, \dots, n.$$

As a consequence, there is a constant $C_0 = C_0(N, \mu, q, p, \Omega, \beta_0) > 1$ such that

$$u(x) \leq C_0 \quad \text{for all } x \in \bigcup_{i=1}^{n} B\Big(x_i, \frac{\beta_0}{32}\Big).$$

Set

$$v(x) := C_1\Big(|x| - \frac{\beta_0}{4}\Big)^{-\frac{2-q}{p+q-1}},$$

where $C_1 > 0$ will be chosen later. We will show that $v(x) \geq u(x)$ for every $x \in \Omega \setminus \mathcal{A}$. Indeed, by a direct computation, we can show that there is a constant $C_1 > 0$ such that, for all $x \in \Omega \setminus \overline{\mathcal{A}}$,

$$\begin{aligned} -\Delta v &= \frac{2-q}{q+p-1}\bigg((N-1)|x|^{-1}\Big(|x| - \frac{\beta_0}{4}\Big)^{-1} - \frac{p+1}{p+q-1}\Big(|x| - \frac{\beta_0}{4}\Big)^{-2}\bigg)v \\ &\geq -C_1\frac{(2-q)(1+p)}{(p+q-1)^2}\Big(|x| - \frac{\beta_0}{4}\Big)^{-\frac{2p+q}{p+q-1}}, \end{aligned} \tag{4.16}$$

$$|\nabla v|^q = C_1^q\Big(\frac{2-q}{p+q-1}\Big)^q\Big(|x| - \frac{\beta_0}{4}\Big)^{-\frac{q(1+p)}{p+q-1}}, \tag{4.17}$$

$$\mu\frac{v(x)}{\delta(x)^2} \leq C_1\varepsilon_0^{-2}(\sup_{x\in\Omega}|x|)^2\Big(|x| - \frac{\beta_0}{4}\Big)^{-\frac{2p+q}{p+q-1}}. \tag{4.18}$$

Gathering estimates (4.16)–(4.18) leads to, for $C_1 = C_1(N, \mu, p, q, \beta_0, \Omega) > 0$ large enough,

$$\begin{aligned} -L_\mu v + v^p|\nabla v|^q &\geq \Big(|x| - \frac{\beta_0}{4}\Big)^{-\frac{2p+q}{p+q-1}}\bigg[-C_1\frac{(2-q)(1+p)}{(p+q-1)^2} - C_1\varepsilon_0^{-2}(\sup_{x\in\Omega}|x|)^2 + C_1^{p+q}\Big(\frac{2-q}{p+q-1}\Big)^q\bigg] \\ &\geq 0 \quad \text{for all } x \in \Omega \setminus \overline{\mathcal{A}}. \end{aligned}$$

Moreover, we can choose $C_1 = C_1(C_0, N, \mu, q, p, \Omega, \beta_0)$ large enough such that $\limsup_{x\to\partial(\Omega\setminus\overline{\mathcal{A}})}(u - v) < 0$. By Lemma 4.3, we deduce that $u \leq v$ in $\Omega \setminus \overline{\mathcal{A}}$, which implies that $u \leq C_2$ in D_{β_0} for some positive constant $C_2 = C_2(N, \mu, q, p, \Omega, \beta_0)$. Thus, by Lemma 4.5, there exists $C_3 = C_3(\Omega, N, \mu, q, p, \beta_0) > 0$ such that

$$u(x) \leq C_3\delta(x)^{-\frac{2-q}{p+q-1}} \quad \text{for all } x \in \Omega.$$

Step 2. For $\ell > 0$, put

$$T_\ell[u](x) := \ell^{\frac{2-q}{p+q-1}} u(\ell x), \quad x \in \Omega^\ell := \ell^{-1}\Omega. \tag{4.19}$$

Let $\xi \in \partial\Omega \setminus \{0\}$, and put $d = d(\xi) := \frac{1}{2}|\xi|$. We assume that $d \le 1$. Denote $u_d := T_d[u]$. Then u_d is a solution of (4.2) in $\Omega^d = \frac{1}{d}\Omega$. Let $R_0 = \frac{\beta_0}{16}$, where β_0 is the constant in Proposition 2.6. Then the solution $w_{\xi,\frac{3R_0}{4}}$ mentioned in Proposition A.1 satisfies $u_d(y) \le w_{\xi,\frac{3R_0}{4}}(y)$ for all $y \in B_{\frac{3R_0}{4}}(\xi) \cap \Omega^d$. Thus u_d is bounded above in $B_{\frac{3R_0}{5}}(\xi) \cap \Omega^d$ by a constant $C > 0$ depending only on N, μ, p, q and the C^2 characteristic of Ω^d (see [29] for the definition of the C^2 characteristic of Ω). As $d \le 1$, a C^2 characteristic of Ω is also a C^2 characteristic of Ω^d; therefore, the constant C can be taken to be independent of ξ. We note here that the constant $R_0 \in (0, 1)$ depends on the C^2 characteristic of Ω. The rest of the proof can proceed similarly to the proof of [16, Proposition E], and we omit it. □

4.3 Weak Singularities

Proof of Theorem 1.6. We use the same idea as in the proof of [16, Theorem F]. Let $u = u^\Omega_{0,k}$ be the positive solution of (1.13). By Theorem 1.3 and Lemma 4.2, $0 < u \le kK_\mu(\cdot, 0)$ in Ω. Moreover,

$$u + \mathbb{G}_\mu[u^p|\nabla u|^q] = kK_\mu(\cdot, 0). \tag{4.20}$$

This and (2.2) imply that

$$u(x) \le kK_\mu(x, 0) \le ck\delta(x)^\alpha |x|^{2-N-2\alpha} \quad \text{for all } x \in \Omega. \tag{4.21}$$

By proceeding as in the proof of (4.13), we obtain

$$|\nabla u(x)| \le ck\delta(x)^{\alpha-1}|x|^{2-N-2\alpha} \quad \text{for all } x \in \Omega. \tag{4.22}$$

It follows from (2.1), (4.21) and (4.22) that

$$\mathbb{G}_\mu[u^p|\nabla u|^q](x) \le ck^{p+q} \int_\Omega \delta(y)^{\alpha p+(\alpha-1)q} G_\mu(x, y)|y|^{(2-N-2\alpha)(p+q)}\, dy. \tag{4.23}$$

Case 1: $\alpha + \alpha p + (\alpha - 1)q \ge 0$. By the assumption and (2.1), we have

$$\mathbb{G}_\mu[u^p|\nabla u|^q](x) \le ck^{p+q}\delta(x)^\alpha \int_\Omega |x-y|^{2-N-2\alpha}|y|^{\alpha-(N+\alpha-2)p-(N+\alpha-1)q}\, dy. \tag{4.24}$$

Since p and q satisfy (1.11), it follows that

$$\int_\Omega |x-y|^{2-N-2\alpha}|y|^{\alpha-(N+\alpha-2)p-(N+\alpha-1)q}\, dy \le c|x|^{2-\alpha-(N+\alpha-2)p-(N-1+\alpha)q}. \tag{4.25}$$

Combining (4.24), (4.25) and (2.2) yields

$$\mathbb{G}_\mu[u^p|\nabla u|^q](x) \le ck^{p+q}|x|^{N+\alpha-(N+\alpha-2)p-(N+\alpha-1)q}K_\mu(x, 0).$$

As a consequence,

$$\lim_{|x|\to 0} \frac{\mathbb{G}_\mu[u^p|\nabla u|^q](x)}{K_\mu(x, 0)} = 0. \tag{4.26}$$

Case 2: $-1 + \alpha < \alpha + \alpha p + (\alpha - 1)q < 0$. By (4.23) and (2.1), we have

$$\mathbb{G}_\mu[u^p|\nabla u|^q](x) \le ck^{p+q} \int_\Omega \delta(y)^{\alpha p+(\alpha-1)q} F_\mu(x, y)|y|^{(2-N-2\alpha)(p+q)}\, dy,$$

where

$$F_\mu(x, y) := |x-y|^{2-N} \min\{1, \delta(x)^\alpha\delta(y)^\alpha|x-y|^{-2\alpha}\} \quad \text{for all } x, y \in \Omega,\ x \ne y. \tag{4.27}$$

Let $\beta \in (0, \beta_0)$ be such that $\delta \in C^2(\overline{\Omega_\beta})$. We consider the cut-off function $\phi \in C^\infty(\overline{\Omega_{\frac{\beta}{2}}})$ such that $0 \le \phi \le 1$, $\phi = 1$ in $\Omega_{\frac{\beta}{4}}$ and $\phi = 0$ in $\Omega \setminus \overline{\Omega_{\frac{\beta}{2}}}$. Then

$$\begin{aligned}\int_\Omega \delta(y)^{\alpha p+(\alpha-1)q} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}\,dy \\ = \int_\Omega \delta(y)^{\alpha p+(\alpha-1)q} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy \\ + \int_\Omega \delta(y)^{\alpha p+(\alpha-1)q} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}(1-\phi(y))\,dy.\end{aligned} \tag{4.28}$$

We first deal with the first term on the right-hand side of (4.28). By the definition of ϕ and the inequality (which follows from (4.27))

$$F_\mu(x,y) \le \delta(x)^\alpha |x-y|^{2-N-\alpha},$$

we deduce that there exists $C = C(N, \mu, p, q, \Omega, \beta)$ such that

$$\int_\Omega \delta(y)^{\alpha p+(\alpha-1)q} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}(1-\phi(y))\,dy \le C\delta(x)^\alpha. \tag{4.29}$$

Now we deal with the second term on the right-hand side of (4.28). Let $\tilde\beta \in (0, \frac{\beta}{4})$ be such that $|x-y| > r_0 > 0$ for any $y \in \Omega_{\tilde\beta}$ and some $r_0 > 0$. Let $\varepsilon > 0$ be such that

$$(N+\alpha-2)p + (N+\alpha-1)q = N+\alpha-\varepsilon,$$

and let $\tilde\varepsilon \in (0,\varepsilon)$ be such that $\alpha p + (\alpha-1)q + 1 - \tilde\varepsilon > 0$. Then, by (4.27), we have

$$\int_{\Sigma_{\tilde\beta}} \delta(y)^{\alpha p+(\alpha-1)q+1} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}\,dS(y) \le \delta(x)^\alpha r_0^{2-N-2\alpha} \int_{\Sigma_{\tilde\beta}} \delta(y)^{\tilde\varepsilon}|y|^{-N+1+(\varepsilon-\tilde\varepsilon)}\,dS(y).$$

Note that, by the choice of $\tilde\varepsilon$, $N-2-N+1+(\varepsilon-\tilde\varepsilon) > -1$, which implies

$$\sup_{\tilde\beta\in(0,\frac{\beta}{4})} \int_{\Sigma_{\tilde\beta}} |y|^{-N+1+(\varepsilon-\tilde\varepsilon)}\,dS(y) < C.$$

Combining the above estimates, we deduce

$$\lim_{\tilde\beta\to 0} \int_{\Sigma_{\tilde\beta}} \delta(y)^{\alpha p+(\alpha-1)q+1} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}\,dS(y) = 0. \tag{4.30}$$

Now note that

$$\begin{aligned}-\int_{\Omega_\beta} \nabla\delta(y)\nabla_y F_\mu(x,y)\delta(y)^{\alpha p+(\alpha-1)q+1}|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy \\ = (N-2)\int_{\Omega_\beta} \frac{\nabla\delta(y)\cdot(x-y)}{|x-y|^N}\min\Big\{1, \frac{\delta(x)^\alpha\delta(y)^\alpha}{|x-y|^{2\alpha}}\Big\}\delta(y)^{\alpha p+(\alpha-1)q+1}|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy \\ - \int_{\Omega_\beta} \nabla\delta(y)\nabla_y\Big(\min\Big\{1, \frac{\delta(x)^\alpha\delta(y)^\alpha}{|x-y|^{2\alpha}}\Big\}\Big)\delta(y)^{\alpha p+(\alpha-1)q+1}|x-y|^{2-N}|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy.\end{aligned}$$

On the other hand,

$$-\nabla\delta(y)\nabla_y(\min\{1, \delta(x)^\alpha\delta(y)^\alpha|x-y|^{-2\alpha}\}) \le 2\alpha|x-y|^{-1}\min\{1, \delta(x)^\alpha\delta(y)^\alpha|x-y|^{-2\alpha}\} \quad \text{a.e. in } \Omega.$$

By collecting the above estimates, we obtain

$$\begin{aligned}-\int_{\Omega_\beta} \nabla\delta(y)\nabla F_\mu(x,y)\delta(y)^{\alpha p+(\alpha-1)q+1}|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy \\ \le C\delta(x)^\alpha \int_\Omega |x-y|^{-(N+\alpha-1)}|y|^{-(N+\alpha-2)p-(N+\alpha-1)q+1}\,dy.\end{aligned} \tag{4.31}$$

It follows from integration by parts, (4.30) and (4.31) that

$$\begin{aligned}
&\int_{\Omega_\beta} \delta(y)^{\alpha p+(\alpha-1)q} F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy \\
&\quad= \frac{1}{\alpha p+(\alpha-1)q+1}\int_{\Omega_\beta} \nabla(\delta(y)^{\alpha p+(\alpha-1)q+1})\nabla\delta(y)F_\mu(x,y)|y|^{-(N+2\alpha-2)(p+q)}\phi(y)\,dy \\
&\quad\le C\delta(x)^\alpha \int_\Omega |x-y|^{-(N+2\alpha-2)}|y|^{-(N+\alpha-2)p-(N+\alpha-1)q+\alpha}\,dy \\
&\qquad+ C\delta(x)^\alpha \int_\Omega |x-y|^{-(N+\alpha-1)}|y|^{-(N+\alpha-2)p-(N+\alpha-1)q+1}\,dy \\
&\quad=: M(x)+N(x).
\end{aligned} \tag{4.32}$$

Since $0<\alpha<1$, $N\ge 3$ and p and q satisfy (1.11), we infer from (2.2) that

$$\max\{M(x),N(x)\} \le C|x|^{N+\alpha-(N+\alpha-2)p-(N+\alpha-1)q}K_\mu(x,0). \tag{4.33}$$

Combining (4.23), (4.28), (4.29), (4.32) and (4.33) implies that there exists a positive constant

$$C = C(N,\mu,p,q,\Omega) > 0$$

such that

$$\mathbb{G}_\mu[u^p|\nabla u|^q](x) \le Ck^q|x|^{N+\alpha-(N+\alpha-2)p-(N+\alpha-1)q}K_\mu(x,0) \quad \text{for all } x\in\Omega. \tag{4.34}$$

Since p and q satisfy (1.11), we deduce (4.26) from (4.34).

Thus, from (4.26) and (4.20), we obtain (1.14). Finally, the monotonicity comes from the comparison principle. □

4.4 Strong Singularities

Let S^{N-1} be the unit sphere in $\mathbb{R}^N$ and $\mathbb{R}^N_+ = \{x=(x_1,\dots,x_N)=(x',x_N) : x_N>0\}$. We denote by

$$x=(r,\sigma)\in\mathbb{R}_+\times S^{N-1} \quad \text{with } r=|x| \text{ and } \sigma = r^{-1}x$$

the spherical coordinates in $\mathbb{R}^N$, and we recall the representation

$$\nabla u = u_r\mathbf{e} + \frac{1}{r}\nabla' u, \quad \Delta u = u_{rr} + \frac{N-1}{r}u_r + \frac{1}{r^2}\Delta' u,$$

where ∇' denotes the covariant derivative on S^{N-1} identified with the tangential derivative and Δ' is the Laplace–Beltrami operator on S^{N-1}.

We look for a particular positive solution of

$$\begin{cases} -L_\mu u + |u|^p|\nabla u|^q = 0 & \text{in } \mathbb{R}^N_+, \\ u=0 & \text{on } \partial\mathbb{R}^N_+\setminus\{0\} = \mathbb{R}^{N-1}\setminus\{0\}, \end{cases}$$

under the separable form

$$u(x) = u(r,\sigma) = r^{-\frac{2-q}{p+q-1}}\omega(\sigma) \quad (r,\sigma)\in(0,\infty)\times S^{N-1}_+.$$

It follows from a straightforward computation that $\omega>0$ satisfies

$$\begin{cases} -\mathcal{L}_\mu\omega - \ell_{N,p,q}\omega + J(\omega,\nabla'\omega) = 0 & \text{in } S^{N-1}_+, \\ \omega = 0 & \text{on } \partial S^{N-1}_+, \end{cases} \tag{4.35}$$

where

$$\mathcal{L}_\mu \omega := \Delta' \omega + \frac{\mu}{(\mathbf{e}_N \cdot \sigma)^2} w, \quad \ell_{N,p,q} := \frac{2-q}{p+q-1}\Big(\frac{2p+q}{p+q-1} - N\Big),$$
$$J(s,\xi) := s^p\Big(\Big(\frac{2-q}{p+q-1}\Big)^2 s^2 + |\xi|^2\Big)^{\frac{q}{2}}, \quad (s,\xi) \in \mathbb{R}_+ \times \mathbb{R}^N.$$

Let κ_μ be the first eigenvalue of $-\mathcal{L}_\mu$ in S_+^{N-1} and ϕ_μ the corresponding eigenfunction $\phi_\mu(\sigma) = (\mathbf{e}_N \cdot \sigma)^\alpha$ for $\sigma \in S_+^{N-1}$, where $\mathbf{e}_N$ is the unit vector pointing toward the north pole.

Notice that the eigenvalue κ_μ is explicitly determined by

$$\kappa_\mu = \alpha(N + \alpha - 2),$$

and the corresponding eigenfunction $\phi_\mu(\sigma) = (\frac{x_N}{|x|}|_{S_+^{N-1}})^\alpha = (\mathbf{e}_N \cdot \sigma)^\alpha$ solves

$$\begin{cases} -\mathcal{L}_\mu \phi_\mu = \kappa_\mu \phi_\mu & \text{in } S_+^{N-1}, \\ \phi_\mu = 0 & \text{on } \partial S_+^{N-1}. \end{cases} \tag{4.36}$$

Notice that equation (4.36) admits a unique positive solution with supremum 1, and if $\mu = 0$, then $\alpha = 1$, which means that $\phi_0(\sigma) = \mathbf{e}_N \cdot \sigma$ is the first eigenfunction of $-\Delta'$ in $H_0^1(S_+^{N-1})$.

We could have defined the first eigenvalue κ_μ of the operator $\mathcal{L}_\mu$ by

$$\kappa_\mu = \inf\Bigg\{\frac{\int_{S_+^{N-1}}(|\nabla' w|^2 - \mu(\mathbf{e}_N \cdot \sigma)^{-2} w^2)\, dS}{\int_{S_+^{N-1}} w^2\, dS} : w \in H_0^1(S_+^{N-1}),\ w \neq 0\Bigg\}.$$

By [12, Theorem 6.1], the infimum exists since $\phi_0(\sigma) = \mathbf{e}_N \cdot \sigma$ is the first eigenfunction of $-\Delta'$ in $H_0^1(S_+^{N-1})$. The minimizer ϕ_μ belongs to $H_0^1(S_+^{N-1})$ only if $1 < \mu < \frac{1}{4}$.

By (4.36), the following expression holds:

$$|\nabla' \phi_0(\sigma)|^2 = 1 - \phi_0(\sigma)^2 \quad \text{for all } \sigma \in S_+^{N-1}. \tag{4.37}$$

Indeed, since $\phi_{\frac{1}{4}} = \phi_0^{\frac{1}{2}}$, we have

$$-\Delta' \phi_0^{\frac{1}{2}} = \frac{1}{4}\phi_0^{-\frac{3}{2}}|\nabla' \phi_0|^2 + \frac{N-1}{2}\phi_0^{\frac{1}{2}} = \frac{1}{4}\phi_0^{-\frac{3}{2}} + \kappa_{\frac{1}{4}}\phi_0^{\frac{1}{2}}.$$

Taking into account that $\kappa_{\frac{1}{4}} - \frac{N-1}{2} = -\frac{1}{4}$, from the above equalities, we obtain (4.37).

Denote

$$\mathbf{Y}_\mu(S_+^{N-1}) := \{\phi \in H^1_{\mathrm{loc}}(S^{N-1}) : \phi_0^{-\alpha}\phi \in H^1(S_+^{N-1}, \phi_0^{2\alpha})\}.$$

It is asserted below that condition (1.11) is sharp for the existence of a positive solution of (4.35).

Theorem 4.7. *Assume $p \geq 0$, $q \geq 0$ and $p + q > 1$.*
(i) *If* (1.11) *does not hold, then there exists no positive solution of* (4.35).
(ii) *If* (1.11) *holds and $q \geq 1$, then problem* (4.35) *admits a unique positive solution $\omega \in \mathbf{Y}_\mu(S_+^{N-1})$. Moreover, there exists a positive constant $C = C(N, \mu, p, q)$ such that*

$$\omega(\sigma) \leq \Big(\frac{\ell_{N,p,q} - \kappa_\mu}{\alpha^q}\Big)^{\frac{1}{p+q-1}}\phi_\mu(\sigma) \quad \textit{for all } \sigma \in S_+^{N-1},$$
$$|\nabla' \omega(\sigma)| \leq C\phi_\mu(\sigma)^{\frac{\alpha-1}{\alpha}} \quad \textit{for all } \sigma \in S_+^{N-1}.$$

Proof. (i) By multiplying (4.35) by ϕ_μ, we obtain

$$(\kappa_\mu - \ell_{N,p,q})\int_{S_+^{N-1}} \omega\phi_\mu\, d\sigma + \int_{S_+^{N-1}} J(\omega, \nabla'\omega))\phi_\mu\, d\sigma = 0. \tag{4.38}$$

Note that the second term on the left-hand side of (4.38) is nonnegative. Thus, if (1.11) does not hold, or equivalently $\ell_{N,p,q} \leq \kappa_\mu$, then no positive solution of (4.35) exists.

(ii) The proof is split into two steps.

Step 1: Existence. Set

$$\gamma_1 := \Big(\frac{\ell_{N,p,q} - \kappa_\mu}{\alpha^q}\Big)^{\frac{1}{p+q-1}}.$$

Then the function $\overline{\omega} = \gamma_1 \phi_\mu$ is a supersolution of (4.35). Indeed, by (4.36) and (4.37),

$$\begin{aligned}
-\mathcal{L}_\mu \overline{\omega} - \ell_{N,p,q}\overline{\omega} + J(\overline{\omega}, \nabla'\overline{\omega}) &= \gamma_1(\kappa_\mu - \ell_{N,p,q})\phi_\mu + \gamma_1^{p+q}\phi_\mu^p\Big(\Big(\frac{2-q}{p+q-1}\Big)^2\phi_\mu^2 + |\nabla'\phi_\mu|^2\Big)^{\frac{q}{2}} \\
&= \gamma_1(\kappa_\mu - \ell_{N,p,q})\phi_0^\alpha + \gamma_1^{p+q}\phi_0^{\alpha p}\Big(\Big(\Big(\frac{2-q}{p+q-1}\Big)^2 - \alpha^2\Big)\phi_0^{2\alpha} + \alpha^2\phi_0^{2(\alpha-1)}\Big)^{\frac{q}{2}} \\
&\geq \gamma_1(\kappa_\mu - \ell_{N,p,q})\phi_0^\alpha + \alpha^q\gamma_1^{p+q}\phi_0^{\alpha p + q(\alpha-1)} \\
&\geq [\gamma_1(\kappa_\mu - \ell_{N,p,q}) + \alpha^q\gamma_1^{p+q}]\phi_0^\alpha = 0.
\end{aligned}$$

In the above estimates, we note that (1.11) implies $\frac{2-q}{p+q-1} > \alpha$.

Let $\alpha_0 \in (\alpha, 1)$ be such that

$$q < \frac{N+\alpha_0}{N+\alpha_0-1} < q_\mu.$$

We note that $\phi_{\mu_0} = \phi_0^{\alpha_0}$, where $\mu_0 = \frac{1}{4} - (\alpha_0 - \frac{1}{2})^2 < \mu$.

We allege that there exists a positive constant $\gamma_2 = \gamma_2(N, q, \mu, \mu_0) \leq \gamma_1$ such that the function $\underline{\omega} = \gamma_2\phi_{\mu_0}$ is a subsolution of (4.35). Indeed, since $q \geq 1$, by (4.36) and (4.37), we have

$$\begin{aligned}
&-\mathcal{L}_\mu\underline{\omega} - \ell_{N,p,q}\underline{\omega} + J(\underline{\omega}, \nabla'\underline{\omega}) \\
&\quad = \gamma_2(\mu_0 - \mu)\frac{\phi_{\mu_0}}{(\mathbf{e}_N\cdot\sigma)^2} + \gamma_2(\kappa_{\mu_0} - \ell_{N,p,q})\phi_{\mu_0} + \gamma_2^{p+q}\phi_{\mu_0}^p\Big(\Big(\frac{2-q}{p+q-1}\Big)^2\phi_{\mu_0}^2 + |\nabla'\phi_{\mu_0}|^2\Big)^{\frac{q}{2}} \\
&\quad \leq (\gamma_2(\mu_0-\mu) + \gamma_2^{q+p}\alpha_0^q)\phi_0^{\alpha_0-2} + \Big(\gamma_2(\kappa_{\mu_0} - \ell_{N,p,q}) + \gamma_2^{q+p}\Big|\Big(\frac{2-q}{p+q-1}\Big)^2 - \alpha_0^2\Big|^{\frac{q}{2}}\Big)\phi_0^{\alpha_0} \leq 0,
\end{aligned}$$

provided γ_2 is small enough. Notice that we can choose $\gamma_2 \leq \gamma_1$.

For $t \in (0,1)$, set $S_t := \{\sigma \in S_+^{N-1} : \phi_0(\sigma) < t\}$, $\widetilde{S}_t := S_+^{N-1} \setminus S_t$. In view of the proof of [20, Theorem 6.5], there exists a solution $\omega_t \in W^{2,p}(\widetilde{S}_t)$ to (4.35) such that

$$\underline{\omega}(\sigma) \leq \omega_t(\sigma) \leq \overline{\omega}(\sigma) \quad \text{for all } \sigma \in \widetilde{S}_t. \tag{4.39}$$

Therefore, by the standard elliptic theory, there exist a function $\widetilde{w}$ and a sequence $t_n \searrow 0$ such that $\omega_{t_n} \to \widetilde{\omega}$ locally uniformly in $C^1(S_+^{N-1})$ and $\widetilde{\omega}$ satisfies $-\mathcal{L}_\mu\widetilde{\omega} - \ell_{N,p,q}\widetilde{\omega} + J(\widetilde{\omega}, \nabla'\widetilde{\omega}) = 0$ in S_+^{N-1}. Furthermore, by (4.39), we have $\underline{\omega}(\sigma) \leq \widetilde{\omega}(\sigma) \leq \overline{\omega}(\sigma)$ for all $\sigma \in S_+^{N-1}$.

Set $\tilde{u}(x) = |x|^{-\frac{2-q}{p+q-1}}\widetilde{\omega}(\sigma)$. Then $\tilde{u}$ satisfies $-L_\mu\tilde{u} + \tilde{u}^p|\nabla\tilde{u}|^q = 0$ in $\mathbb{R}_+^N$ and

$$|\tilde{u}(x)| \leq \Big(\frac{\ell_{N,p,q} - \kappa_\mu}{\alpha^q}\Big)^{\frac{1}{p+q-1}} x_N^\alpha |x|^{-\frac{2-q}{p+q-1}-\alpha} \quad \text{for all } x \in \mathbb{R}_+^N.$$

Let $x_0 = (x_0', 0)$ be such that $|x_0'| = 1$. Then, in view of the proof of (4.13), there exists a constant $C_1 = C(N, \mu, q)$ such that $|\nabla\tilde{u}(x)| \leq C_1 x_N^{\alpha-1}$ for all $x \in B(x_0, \frac{1}{2})$. This implies

$$|\nabla'\tilde{\omega}(\sigma)| \leq C\phi_0(\sigma)^{\alpha-1} \quad \text{for all } \sigma \in S_+^{N-1}. \tag{4.40}$$

Step 2: Uniqueness. Let $\omega_i \in \mathbf{Y}_\mu(S_+^{N-1})$, $i = 1, 2$, be two positive solutions of (4.35). Let $x_0 = (x_0', 0)$ be such that $|x_0'| = 1$. Put $u_i(x) = |x|^{-\frac{2-q}{p+q-1}}\omega_i$. Then $u_i \in H^1(B(x_0, \frac{2}{3}), x_N^{2\alpha})$, and it satisfies $-L_\mu u_i + u_i^p|\nabla u_i|^q = 0$ in $\mathbb{R}_+^N$, which implies $-L_\mu u_i \leq 0$ in $\mathbb{R}_+^N$.

Since $0 < v_i := x_N^{-\alpha}u_i \in H^1(B(x_0, \frac{1}{2}), x_N^{2\alpha})$, and it satisfies $-\operatorname{div}(x_N^{2\alpha}\nabla v) \leq 0$ in $\mathbb{R}_+^N$, by [14, Theorem 2.12], there exists a positive constant $C_i > 0$ such that $u_i(x) \leq C_i x_N^\alpha$ for all $x \in B(x_0, \frac{1}{2})$. Therefore, in view of the proof of (4.40), there exists a positive constant C_0 such that

$$w_i(\sigma) \leq C_0\phi_0(\sigma)^\alpha \qquad \text{for all } \sigma \in S_+^{N-1},\ i = 1, 2, \tag{4.41}$$

$$|\nabla' w_i(\sigma)| \leq C_0\phi_0(\sigma)^{\alpha-1} \quad \text{for all } \sigma \in S_+^{N-1},\ i = 1, 2. \tag{4.42}$$

Set $b_t := \inf_{c>1}\{c : c\omega_1 \geq \omega_2, \sigma \in \widetilde{S}_t\} < \infty$. Without loss of generality, we may assume that $b_{t_0} > 1$ for some $t_0 \in (0,1)$; thus, by (4.41), we have

$$1 < b_{t_0} \leq b_t \quad \text{for all } t \in (0, t_0).$$

In the sequel, we consider $t \in (0, t_0)$.

Put $\psi := \phi_0^\alpha - \frac{1}{2}\phi_0^{\alpha+\varepsilon}$, where $\varepsilon \in (0, 1-\alpha)$ is a parameter that will be determined later. Then we have $\frac{1}{2}\phi_0^\alpha \leq \psi \leq \phi_0^\alpha$. We recall that $\phi_0^\alpha = \phi_\mu$ and $\phi_0^{\alpha+\varepsilon} = \phi_{\mu_\varepsilon}$, where $\mu_\varepsilon := \frac{1}{4} - (\alpha + \varepsilon - \frac{1}{2})^2$. From the definition of ψ, it is easy to check that

$$-\mathcal{L}_\mu \psi = \frac{\mu - \mu_\varepsilon}{2}\phi_0^{\alpha+\varepsilon-2} + \phi_0^\alpha\Big(\kappa_\mu - \frac{\kappa_{\mu_\varepsilon}}{2}\phi_0^\varepsilon\Big). \tag{4.43}$$

Now let $\omega_t = b_t^{-1}\omega_2$. We remark that ω_t is a subsolution of (4.35) and $\omega_t - \omega_1 \leq 0$ in $\widetilde{S}_t$. Also, we have

$$-\mathcal{L}_\mu(\omega_t - \omega_1)_+ \leq \Big|-\Big(\frac{\omega_1}{\omega_t}\Big)^p J(\omega_t, \nabla'\omega_t) + J(\omega_1, \nabla'\omega_1)\Big| + \ell_{N,p,q}|\omega_t - \omega_1|. \tag{4.44}$$

Since $1 \leq q < 2$, the following inequality holds for any nonnegative number h_1, h_2, k_1, k_2:

$$-(h_1^2 + h_2^2)^{\frac{q}{2}} + (k_1^2 + k_2^2)^{\frac{q}{2}} \leq (h_1^{q-1} + h_2^{q-1} + k_1^{q-1} + k_2^{q-1})(|h_1 - k_1| + |h_2 - k_2|). \tag{4.45}$$

By applying (4.45) with $h_1 = (\frac{2-q}{p+q-1})\omega_t$, $h_2 = |\nabla'\omega_t|$, $k_1 = (\frac{2-q}{p+q-1})\omega_1$ and $k_2 = |\nabla'\omega_1|$ and keeping in mind estimates (4.41) and (4.42), we obtain

$$-\Big(\frac{\omega_1}{\omega_t}\Big)^p J(\omega_t, \nabla'\omega_t) + J(\omega_1, \nabla'\omega_1) \leq C(q, C_0)\phi_0^{\alpha p+(q-1)(\alpha-1)}(|\omega_t - \omega_1| + |\nabla'(\omega_t - \omega_1)|). \tag{4.46}$$

Now set $V_t := \psi^{-1}(\omega_t - \omega_1)_+$. By (4.44), (4.46) and the definition of ψ, we can easily deduce the existence of a positive constant $C = C(N, \mu, q, C_0)$ such that

$$-\operatorname{div}'(\psi^2\nabla' V_t) + \psi V_t(-\mathcal{L}_\mu\psi) \leq C(\phi_0^{\alpha p+q(\alpha-1)+\alpha}|\psi^{-1}(\omega_t - \omega_1)| + \phi_0^{\alpha p+(q-1)(\alpha-1)+2\alpha}|\nabla'(\psi^{-1}(\omega_t - \omega_1))|).$$

Now, since $\psi V_t \in \mathbf{Y}_\mu(S_+^{N-1})$ and $V_t(\sigma) \leq 0$ for any $\sigma \in \widetilde{S}_t$, multiplying the above inequality by $(V_t)_+$ and integrating over S_+^{N-1}, we get

$$\begin{aligned}\int_{S_t} |\nabla'(V_t)_+|^2\psi^2\, dS(\sigma) &+ \int_{S_t} \psi(V_t)_+^2(-\mathcal{L}_\mu\psi)\, dS(\sigma)\\ &\leq C\Big(\int_{S_t} \phi_0^{\alpha p+q(\alpha-1)+\alpha}(V_t)_+^2\, dS(\sigma) + \int_{S_t} \phi_0^{\alpha p+(q-1)(\alpha-1)+2\alpha}|\nabla'(V_t)_+|(V_t)_+\, dS(\sigma)\Big).\end{aligned} \tag{4.47}$$

By the definition of ψ and (4.43), we have

$$\begin{aligned}\int_{S_t} |\nabla'(V_t)_+|^2\psi^2\, dS(\sigma) &+ \int_{S_t} \psi(V_t)_+^2(-\mathcal{L}_\mu\psi)\, dS(\sigma)\\ &\geq \frac{1}{4}\int_{S_t} |\nabla'(V_t)_+|^2\phi_0^{2\alpha}\, dS(\sigma) + \frac{\mu - \mu_\varepsilon}{4}\int_{S_t} (V_t)_+^2\phi_0^{2\alpha+\varepsilon-2}\, dS(\sigma) - \frac{N-1}{2}\int_{S_t} (V_t)_+^2\phi_0^{2\alpha}\, dS(\sigma).\end{aligned} \tag{4.48}$$

Here we note that if $\varepsilon < 1 - \alpha$, then $q < 2 < \frac{2-\alpha-\varepsilon}{1-\alpha}$. This leads to

$$2 - \alpha - \varepsilon - q(1-\alpha) > 0 \quad \text{and} \quad 4 - 2\alpha - \varepsilon - 2q(1-\alpha) > 0. \tag{4.49}$$

By Young's inequality, we deduce that

$$\begin{aligned}C\int_{S_t} &\phi_0^{\alpha p+(q-1)(\alpha-1)+2\alpha}|\nabla'(V_t)_+|(V_t)_+\, dS(\sigma)\\ &\leq \frac{1}{8}\int_{S_t} \phi_0^{2\alpha}|\nabla'(V_t)_+|^2\, dS(\sigma) + \hat{C}\int_{S_t} \phi_0^{2\alpha p+2(q-1)(\alpha-1)+2\alpha}(V_t)_+^2\, dS(\sigma),\end{aligned} \tag{4.50}$$

where C is the constant in (4.47) and $\hat{C} = \hat{C}(N, \mu, p, q)$.

Gathering (4.47), (4.48) and (4.50) yields

$$\begin{aligned}\frac{1}{8}\int_{S_t} \phi_0^{2\alpha}|\nabla'(V_t)_+|^2\,dS(\sigma) &\le -\frac{\mu-\mu_\varepsilon}{4}\int_{S_t}\phi_0^{2\alpha+\varepsilon-2}(V_t)_+^2\,dS(\sigma) + C_1\int_{S_t}(\phi_0^{\alpha p+q(\alpha-1)+\alpha} + \phi_0^{2\alpha p+2(q-1)(\alpha-1)+2\alpha} + \phi_0^{2\alpha})(V_t)_+^2\,dS(\sigma)\\ &\le \int_{S_t}\phi_0^{2\alpha+\varepsilon-2}\Big(\frac{\mu_\varepsilon-\mu}{4} + C_1(t^{2+\alpha p-\alpha-\varepsilon-q(1-\alpha)} + t^{4+2\alpha p-2\alpha-\varepsilon-2q(1-\alpha)} + t^{2-\varepsilon})\Big)(V_t)_+^2\,dS(\sigma),\end{aligned}$$

where $C_1 = C(N,\mu,p,q)$. By (4.49) and the above inequality, we can find a positive constant

$$t_1 = t_1(N,q,\mu,\varepsilon,C_0) \quad \text{such that} \quad \frac{1}{8}\int_{S_{t_1}}\phi_0^{2\alpha}|\nabla'(V_{t_1})_+|^2\,dS(\sigma) \le 0,$$

which implies $(V_{t_1})_+ = 0$ in S_{t_1} since $(V_{t_1})_+ = 0$ on $\{\sigma \in S_+^{N-1} : \phi_0(\sigma) = t_1\}$. Hence $b_{t_1}^{-1}\omega_2 \le \omega_1$ for all $\sigma \in S_{t_1}$.

Thus we have proved that

$$b_{t_1} = \inf_{c>1}\{c : c\omega_1 \ge \omega_2,\ \sigma \in \widetilde{S}_{t_1}\} = \inf_{c>1}\{c : c\omega_1 \ge \omega_2,\ \sigma \in S_+^{N-1}\}.$$

This means that $(\omega_1 - \omega_{t_1})(\sigma) \ge 0$ for any $\sigma \in S_+^{N-1}$ and

$$\omega_1(\sigma_0) - \omega_{t_1}(\sigma_0) = 0 \quad \text{for some } \sigma_0 \in \widetilde{S}_{t_1}. \tag{4.51}$$

But $-\mathcal{L}_\mu(\omega_1-\omega_{t_1}) - \ell_{N,q}(\omega_1-\omega_{t_1}) + J(\omega_1,\nabla'\omega_1) - J(\omega_{t_1},\nabla'\omega_{t_1}) \ge 0$, which implies

$$-\Delta'(\omega_1-\omega_{t_1}) + J(\omega_1,\nabla'\omega_1) - J(\omega_t,\nabla'\omega_1) + J(\omega_t,\nabla'\omega_1) - J(\omega_{t_1},\nabla'\omega_{t_1}) \ge 0.$$

By the above inequality, the fact that $\min(\omega_1,\omega_t) > 0$ in $\widetilde{S}_{\frac{t_1}{2}}$ and the mean value theorem, there exists $\bar\Lambda > 0$ such that

$$-\Delta'(\omega_1-\omega_{t_1}) + \frac{\partial J(\bar s,\bar\xi)}{\partial\xi}(\nabla'\omega_1 - \nabla'\omega_{t_1}) + \bar\Lambda(\omega_1-\omega_{t_1}) \ge 0 \quad \text{in } \widetilde{S}_{\frac{t_1}{2}},$$

where $\bar s$ and $\bar\xi$ are functions of $\sigma \in \widetilde{S}_{\frac{t_1}{2}}$ such that $\frac{\partial J(\bar s,\bar\xi)}{\partial\xi} \in L^\infty(\widetilde{S}_{\frac{t_1}{2}})$. By the maximum principle, $\omega_1 - \omega_{t_1}$ cannot achieve a nonpositive minimum in $\widetilde{S}_{\frac{t_1}{2}} \setminus \partial\widetilde{S}_{\frac{t_1}{2}}$, which clearly contradicts (4.51).

The result follows by exchanging the role of ω_1 and ω_2. □

5 Absorption $g(u,|\nabla u|) = |u|^p|\nabla u|^q$: Supercritical Case

Let us recall the following result in [16, 25].

Proposition 5.1. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$, and let β_0 be the constant in Proposition 2.6. Then the following inequalities hold:*

$$\sup_{0<\beta\le\beta_0}\beta^{\alpha-1}\int_{\Sigma_\beta}\mathbb{K}_\mu[\nu]\,dS \le C(\beta_0,\alpha,\Omega)\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \text{if } \mu < \frac{1}{4},$$

$$\sup_{0<\beta\le\beta_0}(\beta|\log\beta|^2)^{-\frac{1}{2}}\int_{\Sigma_\beta}\mathbb{K}_\mu[\nu]\,dS \le C(\beta_0,\alpha,\Omega)\|\nu\|_{\mathfrak{M}(\partial\Omega)} \quad \text{if } \mu = \frac{1}{4}.$$

Lemma 5.2. *Assume $\nu \in \mathfrak{M}^+(\partial\Omega)$, $p \ge 0$, $1 \le q < 2$, and let $u \in C^2(\Omega)$ be a nonnegative solution of* (4.1).

(i) *If $q \ne \alpha + 1$, then there exists a constant $\beta_1 = \beta_1(N,\mu,p,q,\Omega) > 0$ such that*

$$\int_\Omega \delta^{\alpha-q}u^{p+q}\,dx \le C\Big(\int_\Omega \delta^\alpha u^p|\nabla u|^q\,dx + 1\Big), \tag{5.1}$$

where C depends only on N, μ, p, q, Ω and $\sup_{\Sigma_{\beta_1}}(\mathbb{K}_\mu[\nu])^{p+q}$.

(ii) *If $q = \alpha + 1$, then, for any $\varepsilon > 0$ small enough, there exists a constant $\beta_1 = \beta_1(N, \mu, p, \Omega, \varepsilon) > 0$ such that*

$$\int_\Omega \delta^{\varepsilon-1} u^{p+\alpha+1}\,dx \le C\Big(\int_\Omega \delta^\alpha u^p |\nabla u|^{\alpha+1}\,dx + 1\Big), \tag{5.2}$$

where C depends only on N, μ, p, Ω, ε and $\sup_{\Sigma_{\beta_1}}(\mathbb{K}_\mu[\nu])^{p+\alpha+1}$.

Proof. Since u is a nonnegative solution of (4.1) we have $u^{\frac{p}{q}}|\nabla u| \in L^q(\Omega, \delta^\alpha)$. Let $\beta_1 \in (0, \beta_0)$, where β_0 is the constant in Proposition 2.6.

(i) First we assume that $q > 1$, $q \ne \alpha + 1$, and let $\gamma \ne -1$. Then, for $\beta \in (0, \beta_1)$,

$$\begin{aligned}
\int_{D_\beta\setminus D_{\beta_1}} \delta^\gamma u^{p+q}\,dx &= (\gamma+1)^{-1} \int_{D_\beta\setminus D_{\beta_1}} \nabla\delta^{\gamma+1}\nabla\delta u^{p+q}\,dx \\
&= (\gamma+1)^{-1}\Big(-\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}\Delta\delta u^{p+q}\,dx - (p+q)\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q-1}\nabla\delta\nabla u\,dx \\
&\qquad + \int_{\Sigma_{\beta_1}} \delta^{\gamma+1}\frac{\partial\delta}{\partial\mathbf{n}_{\beta_1}} u^{p+q}\,dx + \int_{\Sigma_\beta} \delta^{\gamma+1}\frac{\partial\delta}{\partial\mathbf{n}_\beta} u^{p+q}\,dx\Big) \\
&\le C|\gamma+1|^{-1}\Big(\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q}\,dx + \int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q-1}|\nabla u|\,dx \\
&\qquad + \beta_1^{\gamma+1}\sup_{\Sigma_{\beta_1}}(\mathbb{K}_\mu[\nu])^{p+q} + \int_{\Sigma_\beta} \delta^{\gamma+1}u^{p+q}\,dx\Big).
\end{aligned}$$

Observe that, for any $\gamma \in (\alpha - q, \max\{\frac{\alpha-1-q}{2}, 2(\alpha-q)+1\})$, we have $|\gamma+1|^{-1} < 2|\alpha+1-q|^{-1}$. Therefore, for such γ, we can choose $\beta_1 = \beta_1(N, q, \mu, \Omega)$ such that

$$C|\gamma+1|^{-1}\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q}\,dx \le 2C|\alpha+1-q|^{-1}\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q}\,dx \le \frac{1}{4}\int_{D_\beta\setminus D_{\beta_1}} \delta^\gamma u^{p+q}\,dx.$$

Consequently, by Young's inequality, we can find a constant $C_1 = C_1(N, \mu, p, q, \Omega)$ such that

$$\begin{aligned}
C|\gamma+1|^{-1}\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q-1}|\nabla u|\,dx &= C|\gamma+1|^{-1}\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+1}u^{p+q-1-\frac{p}{q}}u^{\frac{p}{q}}|\nabla u|\,dx \\
&\le \frac{1}{4}\int_{D_\beta\setminus D_{\beta_1}} \delta^\gamma u^{p+q}\,dx + C_1\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+q}u^p|\nabla u|^q\,dx.
\end{aligned}$$

By the above estimates, there is a positive constant $C_2 = C_2(N, \mu, p, q, \Omega)$ such that

$$\int_{D_\beta\setminus D_{\beta_1}} \delta^\gamma u^{p+q}\,dx \le C_2\Big(\int_{D_\beta\setminus D_{\beta_1}} \delta^{\gamma+q}u^p|\nabla u|^q\,dx + \beta_1^{\gamma+1}\sup_{\Sigma_{\beta_1}}(\mathbb{K}_\mu[\nu])^q + \int_{\Sigma_\beta}\delta^{\gamma+1}u^{p+q}\,dx\Big). \tag{5.3}$$

By (4.12), Proposition 5.1 and taking into account that $\gamma + q - 1 > \alpha - 1$, we obtain

$$\int_{\Sigma_\beta} \delta^{\gamma+1}u^{p+q}\,dS \le C\beta^{\gamma+q-1}\int_{\Sigma_\beta} u\,dS \le C\beta^{\gamma+q-1}\int_{\Sigma_\beta}\mathbb{K}_\mu[\nu]\,dS \to 0 \quad \text{as } \beta \to 0.$$

Therefore, by letting $\beta \to 0$ in (5.3), we obtain

$$\int_{\Omega_{\beta_1}} \delta^\gamma u^{p+q}\,dx \le C_2\Big(\int_{\Omega_{\beta_1}} \delta^{\gamma+q}u^p|\nabla u|^q\,dx + \beta_1^{\gamma+1}\sup_{\Sigma_{\beta_1}}(\mathbb{K}_\mu[\nu])^{p+q}\Big). \tag{5.4}$$

By the dominated convergence theorem, we can send $\gamma \to \alpha - q$ in (5.4) to obtain

$$\int_{\Omega_{\beta_1}} \delta^{\alpha-q}u^{p+q}\,dx \le C_2\Big(\int_{\Omega_{\beta_1}} \delta^\alpha u^p|\nabla u|^q\,dx + \beta_1^{\alpha-q+1}\sup_{\Sigma_{\beta_1}}(\mathbb{K}_\mu[\nu])^{p+q}\Big).$$

This implies (5.1).

The proof of (5.2) follows by arguments similar to the proof of (5.1) (with $\gamma = \varepsilon - 1$) with some modifications, and we omit it. □

We recall below some notations concerning the Besov space (see e.g. [1, 33]). For $\sigma > 0$, $1 \leq \kappa < \infty$, we denote by $W^{\sigma,\kappa}(\mathbb{R}^d)$ the Sobolev space over $\mathbb{R}^d$. If σ is not an integer, the Besov space $B^{\sigma,\kappa}(\mathbb{R}^d)$ coincides with $W^{\sigma,\kappa}(\mathbb{R}^d)$. When σ is an integer, we denote $\Delta_{x,y}f := f(x+y) + f(x-y) - 2f(x)$ and

$$B^{1,\kappa}(\mathbb{R}^d) := \Big\{ f \in L^\kappa(\mathbb{R}^d) : \frac{\Delta_{x,y}f}{|y|^{1+\frac{d}{\kappa}}} \in L^\kappa(\mathbb{R}^d \times \mathbb{R}^d) \Big\}$$

with norm

$$\|f\|_{B^{1,\kappa}} := \Big(\|f\|_{L^\kappa}^\kappa + \int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\int \frac{|\Delta_{x,y}f|^\kappa}{|y|^{\kappa+d}}\,dx\,dy \Big)^{\frac{1}{\kappa}}.$$

Then

$$B^{m,\kappa}(\mathbb{R}^d) := \{ f \in W^{m-1,\kappa}(\mathbb{R}^d) : D_x^\theta f \in B^{1,\kappa}(\mathbb{R}^d) \text{ for all } \theta \in \mathbb{N}^d,\ |\theta| = m-1 \}$$

with norm

$$\|f\|_{B^{m,\kappa}} := \Big(\|f\|_{W^{m-1,\kappa}}^\kappa + \sum_{|\theta|=m-1} \int\limits_{\mathbb{R}^d\times\mathbb{R}^d}\int \frac{|D_x^\theta \Delta_{x,y}f|^\kappa}{|y|^{\kappa+d}}\,dx\,dy \Big)^{\frac{1}{\kappa}}.$$

These spaces are fundamental because they are stable under the real interpolation method developed by Lions and Petree. For $s \in \mathbb{R}$, we defined the Bessel kernel of order s by $G_s(\xi) = \mathcal{F}^{-1}(1+|\cdot|^2)^{-\frac{s}{2}}\mathcal{F}(\xi)$, where $\mathcal{F}$ is the Fourier transform of moderate distributions in $\mathbb{R}^d$. The Bessel space $L_{s,\kappa}(\mathbb{R}^d)$ is defined by

$$L_{s,\kappa}(\mathbb{R}^d) := \{ f = G_s * g : g \in L^\kappa(\mathbb{R}^d) \}$$

with norm $\|f\|_{L_{s,\kappa}} := \|g\|_{L^\kappa} = \|G_{-s} * f\|_{L^\kappa}$. It is known that if $1 < \kappa < \infty$ and $s > 0$, $L_{s,\kappa}(\mathbb{R}^d) = W^{s,\kappa}(\mathbb{R}^d)$ if $s \in \mathbb{N}$, and $L_{s,\kappa}(\mathbb{R}^d) = B^{s,\kappa}(\mathbb{R}^d)$ if $s \notin \mathbb{N}$, always with equivalent norms. The Bessel capacity is defined for compact subsets $K \subset \mathbb{R}^d$ by

$$C_{s,\kappa}^{\mathbb{R}^d}(K) := \inf\{ \|f\|_{L_{s,\kappa}}^\kappa,\ f \in \mathcal{S}'(\mathbb{R}^d),\ f \geq \chi_K \}.$$

It is extended to open sets and then Borel sets by the fact that it is an outer measure.

Proof of Theorem 1.7. Let $\varepsilon \geq 0$, and let $u \in C^2(\Omega)$ be the solution of (4.1). Put $\Sigma = \partial\Omega$. If

$$\eta \in L^\infty(\partial\Omega) \cap B^{1-\alpha+\frac{\alpha+1-q}{p+q}+\frac{\varepsilon}{p+q},(p+q)'}(\partial\Omega),$$

we denote by $H := H[\eta]$ the solution of

$$\begin{cases} \dfrac{\partial H}{\partial s} + \Delta_\Sigma H = 0 & \text{in } (0,\infty)\times\partial\Omega, \\ H(0,\cdot) = \eta & \text{on } \partial\Omega. \end{cases}$$

Let $h \in C^\infty(\mathbb{R}_+)$ be such that $0 \leq h \leq 1$, $h' \leq 0$, $h \equiv 1$ on $[0, \frac{\beta_0}{2}]$, $h \equiv 0$ on $[\beta_0, \infty]$. The lifting we consider is expressed by

$$R[\eta](x) := \begin{cases} H[\eta](\delta^2, \sigma(x))h(\delta) & \text{if } x \in \overline{\Omega}_{\beta_0}, \\ 0 & \text{if } x \in D_{\beta_0}, \end{cases} \tag{5.5}$$

with $x = (\delta, \sigma) = (\delta(x), \sigma(x))$.

Case 1: $q \neq \alpha + 1$. Set $\varepsilon = 0$ and $\zeta = \varphi_\mu R[\eta]^{(p+q)'}$, where φ_μ is the eigenfunction associated to the first eigenvalue λ_μ of $-L_\mu$ in Ω (see Subsection 2.1). By proceeding as in the proof of [17, Lemma 3.8, (3.46)], we deduce that there exists $C_0 = C_0(N, \mu, \Omega, \|\nu\|_{\mathfrak{M}(\partial\Omega)})$ such that

$$\begin{aligned} C_0\Big(\int\limits_{\partial\Omega} \eta\,d\nu\Big)^{(p+q)'} &\leq \int\limits_\Omega u^p|\nabla u|^q \zeta\,dx + \lambda_\mu \int\limits_\Omega u\zeta\,dx \\ &\quad + (p+q)'\Big(\int\limits_\Omega u^{p+q}\varphi_\mu^{-\frac{q}{\alpha}}\zeta\,dx\Big)^{\frac{1}{p+q}}\Big(\int\limits_\Omega L[\eta]^{(p+q)'}\,dx\Big)^{\frac{1}{(p+q)'}}, \end{aligned} \tag{5.6}$$

where

$$L[\eta] := \big(2\varphi_\mu^{\frac{q}{\alpha(p+q)}-\frac{1}{p+q}}|\nabla\varphi_\mu\cdot\nabla R[\eta]| + \varphi_\mu^{1+\frac{q}{\alpha(p+q)}-\frac{1}{p+q}}|\Delta R[\eta]|\big).$$

Following the arguments of the proof of [17, Lemma 3.9, (3.48)], we can obtain

$$\int_\Omega L[\eta]^{(p+q)'}\,dx \le c\|\eta\|_{L^\infty(\partial\Omega)}^{(p+q)'-1}\|\eta\|_{B^{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(\partial\Omega)}. \tag{5.7}$$

We infer from (5.1) that

$$\int_\Omega u^{p+q}\varphi_\mu^{-\frac{q}{\alpha}}\zeta\,dx \le C\|\eta\|_{L^\infty(\partial\Omega)}^{(p+q)'}\int_\Omega \delta^{\alpha-q}u^{p+q}\,dx \le C\|\eta\|_{L^\infty(\partial\Omega)}^{(p+q)'}\Big(1+\int_\Omega u^p|\nabla u|^q\delta^\alpha\,dx\Big), \tag{5.8}$$

where the constant C depends on N,μ,p,q and Ω. Combining (5.6), (5.7) and (5.8), we obtain

$$\begin{aligned} C_0\Big(\int_{\partial\Omega}\eta\,d\nu\Big)^{(p+q)'} &\le \int_\Omega u^p|\nabla u|^q\zeta\,dx + \lambda_\mu\int_\Omega u\zeta\,dx \\ &\quad + C\|\eta\|_{L^\infty(\partial\Omega)}^{\frac{(p+q)'}{p+q}}\Big(1+\int_\Omega u^p|\nabla u|^q\delta^\alpha\,dx\Big)^{\frac{1}{p+q}}\big(\|\eta\|_{L^\infty(\partial\Omega)}^{(p+q)'-1}\|\eta\|_{B^{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(\partial\Omega)}\big)^{\frac{1}{(p+q)'}}. \end{aligned} \tag{5.9}$$

Let $K\subset\partial\Omega$ be a compact set. Since $(N+\alpha-2)p+(N+\alpha-1)q\ge N+\alpha$, if

$$C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(K)=0,$$

then there exists a sequence $\{\eta_n\}$ in $C_0^2(\partial\Omega)$ with the following properties:

$$0\le\eta_n\le 1,\quad \eta_n=1 \text{ in a neighborhood of } K \quad\text{and}\quad \lim_{n\to\infty}\eta_n=0 \text{ in } B^{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(\partial\Omega). \tag{5.10}$$

This implies that $0\le R[\eta_n]\le 1$ and $\lim_{n\to\infty}R[\eta_n]=0$ a.e. in Ω. Put $\zeta_n=\varphi_\mu R[\eta_n]^{(p+q)'}$. Then

$$\lim_{n\to\infty}\int_\Omega u^p|\nabla u|^q\zeta_n\,dx=0 \quad\text{and}\quad \lim_{n\to\infty}\int_\Omega u\zeta_n\,dx=0. \tag{5.11}$$

From (5.9)–(5.11), we obtain

$$\nu(K)\le\int_{\partial\Omega}\eta_n d\nu\to 0 \quad \text{as } n\to\infty.$$

This implies that $\nu(K)=0$. Thus ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}$.

Case 2: $q=\alpha+1$. Let $0<\varepsilon<\alpha+1$ and $\zeta=\varphi_\mu R[\eta]^{(p+\alpha+1)'}$. Proceeding as in the proof of (5.6), we can prove

$$\begin{aligned} C_0\Big(\int_{\partial\Omega}\eta\,d\nu\Big)^{(p+\alpha+1)'} &\le \int_\Omega u^p|\nabla u|^{\alpha+1}\zeta\,dx + \lambda_\mu\int_\Omega u\zeta\,dx \\ &\quad + (p+\alpha+1)'\Big(\int_\Omega u^{p+\alpha+1}\varphi_\mu^{-\frac{\alpha+1-\varepsilon}{\alpha}}\zeta\,dx\Big)^{\frac{1}{p+\alpha+1}}\Big(\int_\Omega L[\eta]^{(p+\alpha+1)'}\,dx\Big)^{(p+1+\alpha)'}, \end{aligned}$$

where

$$L[\eta]=\Big(2\varphi_\mu^{\frac{\alpha+1-\varepsilon}{\alpha(p+\alpha+1)}-\frac{1}{p+\alpha+1}}|\nabla\varphi_\mu\cdot\nabla R[\eta]|+\varphi_\mu^{1+\frac{\alpha+1-\varepsilon}{\alpha(p+\alpha+1)}-\frac{1}{p+\alpha+1}}|\Delta R[\eta]|\Big).$$

Using (5.2) and the ideas of the proof of (5.9), we can obtain the inequality

$$\begin{aligned} C_0\Big(\int_{\partial\Omega}\eta\,d\nu\Big)^{(p+\alpha+1)'} &\le \int_\Omega u^p|\nabla u|^{\alpha+1}\zeta\,dx + \lambda_\mu\int_\Omega u\zeta\,dx \\ &\quad + C\|\eta\|_{L^\infty(\partial\Omega)}^{\frac{(p+\alpha+1)'}{p+\alpha+1}}\Big(1+\int_\Omega u^p|\nabla u|^{\alpha+1}\delta^\alpha\,dx\Big)^{\frac{1}{p+\alpha+1}} \\ &\quad \times\big(\|\eta\|_{L^\infty(\partial\Omega)}^{(p+\alpha+1)'-1}\|\eta\|_{B^{1-\alpha+\frac{\varepsilon}{p+\alpha+1},(p+\alpha+1)'}(\partial\Omega)}\big)^{\frac{1}{(p+\alpha+1)'}}, \end{aligned}$$

where the constant C depends on N,μ,p,Ω and ε.

The rest of the proof follows by using an argument similar to the first case. □

Proposition 5.3. *Let $u \in C^2(\Omega)$ be a positive solution of* (1.19)*. If $u^p|\nabla u| \in L^q(\Omega, \delta^\alpha)$, then u possesses a boundary trace $\nu \in \mathfrak{M}^+(\partial\Omega)$, i.e. u is the solution of boundary value problem* (4.1) *with boundary trace ν.*

Proof. If $v := \mathbb{G}_\mu[u^p|\nabla u|^q]$, then $v \in L^1(\Omega, \delta^\alpha)$ and $u + v$ is a positive L_μ-harmonic function. Hence we have $u + v \in L^1(\Omega, \delta^\alpha)$, and there exists a measure $\nu \in \mathfrak{M}^+(\partial\Omega)$ such that $u + v = \mathbb{K}_\mu[\nu]$. By [16, Proposition 2.2], we obtain the result. □

Proof of Theorem 1.8. In view of the proof of [17, Proposition A.2], we can obtain the estimates

$$
\begin{aligned}
|u(x)| &\le C\delta(x)^\alpha \operatorname{dist}(x, K)^{-\frac{2-q}{p+q-1}-\alpha} && \text{for all } x \in \Omega,\\
|\nabla u(x)| &\le C\delta(x)^{\alpha-1} \operatorname{dist}(x, K)^{-\frac{2-q}{p+q-1}-\alpha} && \text{for all } x \in \Omega,
\end{aligned}
$$

where C depends on N, μ, p, q, Ω and $\sup_{\Sigma_{\beta_0}} u$.

Case 1. Assume that

$$q \neq \alpha + 1 \quad \text{and} \quad C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(K) = 0.$$

Then there exists a sequence $\{\eta_n\}$ in $C^2_0(\partial\Omega)$ satisfying (5.10). In particular, there exists a decreasing sequence $\{\mathcal{O}_n\}$ of relatively open subsets of $\partial\Omega$, containing K such that $\eta_n = 1$ on $\mathcal{O}_n$, and thus $\eta_n = 1$ on $K_n := \overline{\mathcal{O}}_n$. We set

$$\tilde{\eta}_n = 1 - \eta_n \quad \text{and} \quad \tilde{\zeta}_n = \varphi_\mu R[\tilde{\eta}_n]^{2(p+q)'},$$

where R is defined by (5.5). Then $0 \le \tilde{\eta}_n \le 1$ and $\tilde{\eta}_n = 0$ on K_n. Therefore,

$$\tilde{\zeta}_n(x) \le \phi_\mu \min\{1, c\delta(x)^{1-N} e^{-(4\delta(x))^{-2}(\operatorname{dist}(x,K_n^c))^2}\} \quad \text{for all } x \in \Omega.$$

Furthermore,

$$
\begin{aligned}
|\nabla R[\tilde{\eta}_n]| &\le c\min\{1, \delta(x)^{-2-N} e^{-(4\delta(x))^{-2}(\operatorname{dist}(x,K_n^c))^2}\} && \text{for all } x \in \Omega,\\
|\Delta R[\tilde{\eta}_n]| &\le c\min\{1, \delta(x)^{-4-N} e^{-(4\delta(x))^{-2}(\operatorname{dist}(x,K_n^c))^2}\} && \text{for all } x \in \Omega.
\end{aligned}
$$

Proceeding as in the proof of [17, Theorem 3.10, (3.65)], we have

$$\int_\Omega (uL_\mu\tilde{\zeta}_n + u^p|\nabla u|^q\tilde{\zeta}_n)\,dx = 0. \tag{5.12}$$

Using the expression of $L_\mu\tilde{\zeta}_n$, we derive from (5.12) that

$$
\begin{aligned}
\int_\Omega u^p|\nabla u|^q\tilde{\zeta}_n\,dx &= \int_\Omega (-\lambda_\mu\varphi_\mu R[\tilde{\eta}_n]^{2(p+q)'}\\
&\qquad + 4(p+q)'R[\tilde{\eta}_n]^{2(p+q)'-1}\nabla\varphi_\mu\cdot\nabla R[\tilde{\eta}_n]\\
&\qquad + 2(p+q)'R[\tilde{\eta}_n]^{2(p+q)'-2}\varphi_\mu(R[\tilde{\eta}_n]\Delta R[\tilde{\eta}_n] + (2(p+q)'-1)|\nabla R[\tilde{\eta}_n]|^2))u\,dx\\
&\le c\Big(\int_\Omega u^{p+q}\varphi_\mu^{-\frac{q}{\alpha}}\tilde{\zeta}_n\,dx\Big)^{\frac{1}{p+q}}\Big(\int_\Omega \tilde{L}[\eta_n]^{(p+q)'}\,dx\Big)^{\frac{1}{(p+q)'}},
\end{aligned}
$$

where

$$\tilde{L}[\eta] = \varphi_\mu^{\frac{q}{\alpha(p+q)}-\frac{1}{p+q}}|\nabla\varphi_\mu\cdot\nabla R[\eta_n]| + \varphi_\mu^{1+\frac{q}{\alpha(p+q)}-\frac{1}{p+q}}|\Delta R[\tilde{\eta}_n]| + \varphi_\mu^{1+\frac{q}{\alpha(p+q)}-\frac{1}{p+q}}|\nabla R[\tilde{\eta}_n]|^2.$$

By proceeding as in the proof of [17, Theorem 3.10, (3.75)], we can prove

$$\int_\Omega |u|^p|\nabla u|^q\varphi_\mu R[\tilde{\eta}_n]^{2(p+q)'}\,dx \le C\|\eta_n\|_{B^{1-\alpha+\frac{\alpha+1-q}{p+q},(p+q)'}(\partial\Omega)}\Big(\int_\Omega \delta^{\alpha-q}u^q R[\tilde{\eta}_n]^{2q'}\,dx\Big)^{\frac{1}{q}}.$$

The rest of the proof is similar to the proof of [16, Theorem J], and we omit it.

Case 2. Assume that

$$q = \alpha + 1 \quad \text{and} \quad C^{\mathbb{R}^{N-1}}_{1-\alpha+\frac{\varepsilon}{p+\alpha+1},(p+\alpha+1)'}(K) = 0$$

for ε as in statement (ii). Then we can obtain the desired result by combining the ideas in Case 1 of this theorem and in Case 2 of Theorem 1.7. □

6 Nonlinear Equations with Subcritical Source

In this section, we prove Theorem 1.9. We first establish an existence result for the case when g is smooth and bounded.

Lemma 6.1. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$ with $\|\nu\|_{\mathfrak{M}(\partial\Omega)} = 1$ and $g \in C^1(\mathbb{R}\times\mathbb{R}_+) \cap L^\infty(\mathbb{R}\times\mathbb{R}_+)$. Assume* (1.10) *and* (1.21) *are satisfied. Then there exists $\varrho_0 > 0$ depending on $N, \mu, \Omega, \Lambda_g, \tilde{k}$ such that, for every $\varrho \in (0, \varrho_0)$, the problem*

$$\begin{cases} -L_\mu v = g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|) & \text{in } \Omega, \\ \operatorname{tr}(v) = 0 \end{cases} \tag{6.1}$$

admits a positive weak solution v satisfying

$$\|v\|_{L_w^{p_\mu}(\Omega,\delta^\alpha)} + \|\nabla v\|_{L_w^{q_\mu}(\Omega,\delta^\alpha)} \le t_0,$$

where $t_0 > 0$ depends on $N, \mu, \Omega, \Lambda_g, \tilde{k}, \tilde{p}, \tilde{q}$. Here Λ_g is defined in (1.10) *and $\tilde{k}, \tilde{p}, \tilde{q}$ are as in* (1.21).

Proof. We shall use the Schauder fixed point theorem to show the existence of a positive weak solution of (6.1). Define the operator $\mathbb{S}$ by

$$\mathbb{S}(v) := \mathbb{G}_\mu\big[g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)\big], \quad v \in W^{1,1}(\Omega, \delta^\alpha).$$

Fix $1 < \kappa < \min\{\tilde{p}, \tilde{q}, q_\mu\}$,

$$\begin{aligned} Q_1(v) &:= \|v\|_{L_w^{p_\mu}(\Omega,\delta^\alpha)} && \text{for } v \in L_w^{p_\mu}(\Omega, \delta^\alpha), \\ Q_2(v) &:= \|\nabla v\|_{L_w^{q_\mu}(\Omega,\delta^\alpha)} && \text{for } |\nabla v| \in L_w^{q_\mu}(\Omega, \delta^\alpha), \\ Q_3(v) &:= \|v\|_{L^\kappa(\Omega,\delta^\alpha)} && \text{for } v \in L^\kappa(\Omega, \delta^\alpha), \\ Q_4(v) &:= \|\nabla v\|_{L^\kappa(\Omega,\delta^\alpha)} && \text{for } |\nabla v| \in L^\kappa(\Omega, \delta^\alpha) \end{aligned}$$

and

$$Q(v) := Q_1(v) + Q_2(v) + Q_3(v) + Q_4(v).$$

Step 1: Estimate the $L^1(\Omega, \delta^\alpha)$-norm of $g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)$. For $\lambda > 0$ and any function w, we use the notation as in (3.12). For the sake of simplicity, when $w = v + \varrho\mathbb{K}_\mu[\nu]$, we drop the superscript $v + \varrho\mathbb{K}_\mu[\nu]$ in the above notations. For instance, we use the notations $\mathbf{A}_\lambda$ and $\mathbf{a}(\lambda)$ instead of $\mathbf{A}_\lambda^{v+\varrho\mathbb{K}_\mu[\nu]}$ and $\mathbf{a}^{v+\varrho\mathbb{K}_\mu[\nu]}(\lambda)$.

Then, by (2.4), we have

$$\begin{aligned} \mathbf{a}(\lambda) &\le \lambda^{-p_\mu}\|v + \varrho\mathbb{K}_\mu[\nu]\|^{p_\mu}_{L_w^{p_\mu}(\Omega,\delta^\alpha)}, \\ \mathbf{b}(\lambda) &\le \lambda^{-p_\mu}\|\nabla(v + \varrho\mathbb{K}_\mu[\nu])\|^{q_\mu}_{L_w^{q_\mu}(\Omega,\delta^\alpha)}, \\ \mathbf{c}(\lambda) &\le \lambda^{-p_\mu}\min\big\{\|v + \varrho\mathbb{K}_\mu[\nu]\|^{p_\mu}_{L_w^{p_\mu}(\Omega,\delta^\alpha)}, \|\nabla(v + \varrho\mathbb{K}_\mu[\nu])\|^{q_\mu}_{L_w^{q_\mu}(\Omega,\delta^\alpha)}\big\}. \end{aligned}$$

With the above notations, we split

$$\begin{aligned} \|g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)\|_{L^1(\Omega,\delta^\alpha)} \le &\int_{\mathbf{C}_1} g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)\delta^\alpha\,dx \\ &+ \int_{\mathbf{A}_1^c\cap\mathbf{B}_1} g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)\delta^\alpha\,dx \\ &+ \int_{\mathbf{A}_1^c\cap\mathbf{B}_1^c} g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)\delta^\alpha\,dx \\ &+ \int_{\mathbf{A}_1\cap\mathbf{B}_1^c} g(v + \varrho\mathbb{K}_\mu[\nu], |\nabla(v + \varrho\mathbb{K}_\mu[\nu])|)\delta^\alpha\,dx \\ =:&\ I_1 + I_2 + I_3 + I_4. \end{aligned} \tag{6.2}$$

First we estimate I_3. Since $|v + \varrho\mathbb{K}_\mu[v]| \le 1$ and $|\nabla(v + \varrho\mathbb{K}_\mu[v])| \le 1$ in $\mathbf{A}_1^c \cap \mathbf{B}_1^c$ and $1 < \kappa < \min\{\tilde{p}, \tilde{q}, q_\mu\}$, we obtain

$$I_3 \le \tilde{k}\big(\|v + \varrho\mathbb{K}_\mu[v]\|^\kappa_{L^\kappa(\Omega,\delta^\alpha)} + \|\nabla(v + \varrho\mathbb{K}_\mu[v])\|^\kappa_{L^\kappa(\Omega,\delta^\alpha)}\big)g(1,1). \tag{6.3}$$

Next I_1 is estimated as follows:

$$\begin{aligned} I_1 &\le -\int_1^\infty g(\lambda, \lambda^{\frac{p_\mu}{q_\mu}})\,d\mathbf{c}(\lambda) = g(1,1)\mathbf{c}(1) + \int_1^\infty \mathbf{c}(\lambda)\,dg(\lambda, \lambda^{\frac{p_\mu}{q_\mu}}) \\ &\le p_\mu \min\{\|v + \varrho\mathbb{K}_\mu[v]\|^{p_\mu}_{L_w^{p_\mu}(\Omega,\delta^\alpha)}, \|\nabla(v + \varrho\mathbb{K}_\mu[v])\|^{q_\mu}_{L_w^{q_\mu}(\Omega,\delta^\alpha)}\} \int_1^\infty g(\lambda, \lambda^{\frac{p_\mu}{q_\mu}})\lambda^{-1-p_\mu}\,d\lambda. \end{aligned} \tag{6.4}$$

We bound I_4 from above as follows:

$$I_4 \le -\int_1^\infty g(\lambda, 1)\,d\mathbf{a}(\lambda) \le p_\mu\|v + \varrho\mathbb{K}_\mu[v]\|^{p_\mu}_{L_w^{p_\mu}(\Omega,\delta^\alpha)} \int_1^\infty g(\lambda, \lambda^{\frac{p_\mu}{q_\mu}})\lambda^{-1-p_\mu}\,d\lambda. \tag{6.5}$$

Similarly, we can estimate I_2 as follows:

$$I_2 \le p_\mu\|\nabla(v + \varrho\mathbb{K}_\mu[v])\|^{q_\mu}_{L_w^{q_\mu}(\Omega,\delta^\alpha)} \int_1^\infty g(\lambda, \lambda^{\frac{p_\mu}{q_\mu}})\lambda^{-1-p_\mu}\,d\lambda. \tag{6.6}$$

By combining (6.2)–(6.6), we obtain (assuming $\varrho \le 1$)

$$\|g(v + \varrho\mathbb{K}_\mu[v], |\nabla(v + \varrho\mathbb{K}_\mu[v])|)\|_{L^1(\Omega,\delta^\alpha)} \le C\big(Q_1(v)^{p_\mu} + Q_2(v)^{q_\mu} + Q_3(v)^\kappa + Q_4(v)^\kappa + \varrho^\kappa\big), \tag{6.7}$$

where $C = C(N, \mu, \Omega, \tilde{k}, \Lambda_g)$.

Step 2: Estimate Q_1, Q_2, Q_3, Q_4 and Q. By (2.5), we have

$$Q_1(\mathbb{S}(v)) \le c\|g(v + \varrho\mathbb{K}_\mu[v], |\nabla(v + \varrho\mathbb{K}_\mu[v])|)\|_{L^1(\Omega,\delta^\alpha)}.$$

This and (6.7) imply that

$$Q_1(\mathbb{S}(v)) \le C\big(Q_1(v)^{p_\mu} + Q_2(v)^{q_\mu} + Q_3(v)^\kappa + Q_4(v)^\kappa + \varrho^\kappa\big),$$

where $C = C(N, \mu, \Omega, \tilde{k}, \Lambda_g)$. Next we deduce from (2.7) that

$$Q_2(\mathbb{S}(v)) \le c\|g(v + \varrho\mathbb{K}_\mu[v], |\nabla(v + \varrho\mathbb{K}_\mu[v])|)\|_{L^1(\Omega,\delta^\alpha)},$$

which in turn implies

$$Q_2(\mathbb{S}(v)) \le C\big(Q_1(v)^{p_\mu} + Q_2(v)^{q_\mu} + Q_3(v)^\kappa + Q_4(v)^\kappa + \varrho^\kappa\big),$$

where $C = C(N, \mu, \Omega, \tilde{k}, \Lambda_g)$. By (2.3), (2.5) and (2.7), we can easily deduce that

$$\begin{aligned} Q_3(\mathbb{S}(v)) &\le c\|g(v + \varrho\mathbb{K}_\mu[v], |\nabla(v + \varrho\mathbb{K}_\mu[v])|)\|_{L^1(\Omega,\delta^\alpha)}, \\ Q_4(\mathbb{S}(v)) &\le c\|g(v + \varrho\mathbb{K}_\mu[v], |\nabla(v + \varrho\mathbb{K}_\mu[v])|)\|_{L^1(\Omega,\delta^\alpha)}. \end{aligned}$$

Thus,

$$Q_3(\mathbb{S}(v)) + Q_4(\mathbb{S}(v)) \le C\big(Q_1(v)^{p_\mu} + Q_2(v)^{q_\mu} + Q_3(v)^\kappa + Q_4(v)^\kappa + \varrho^\kappa\big),$$

where $C(N, \mu, \Omega, \tilde{k}, \Lambda_g)$. Consequently,

$$Q(\mathbb{S}(v)) \le C\left(Q_1(v)^{p_\mu} + Q_2(v)^{q_\mu} + Q_3(v)^\kappa + Q_4(v)^\kappa + \varrho^\kappa\right).$$

Therefore, if $Q(v) \le t$, then

$$Q(\mathbb{S}(v)) \le C(t^{p_\mu} + t^{q_\mu} + 2t^\kappa + \varrho^\kappa).$$

Since $p_\mu > q_\mu > \kappa > 1$, there exists $\varrho_0 > 0$ depending on $N, \mu, \Omega, \tilde{k}, \Lambda_g$ such that, for any $\varrho \in (0, \varrho_0)$, the equation $C(t^{p_\mu} + t^{q_\mu} + 2t^\kappa + \varrho^\kappa) = t$ admits a largest root $t_0 > 0$ which depends on $N, \mu, \Omega, \Lambda_g, \tilde{k}$. Therefore,

$$Q(v) \le t_0 \implies Q(\mathbb{S}(v)) \le t_0. \tag{6.8}$$

Step 3. We apply the Schauder fixed point theorem to our setting. By a standard argument, we can show that $\mathbb{S}: W^{1,1}(\Omega,\delta^\alpha) \to W^{1,1}(\Omega,\delta^\alpha)$ is continuous and compact. Set

$$\mathcal{O} := \{\xi \in W^{1,1}(\Omega,\delta^\alpha) : Q(u) \le t_0\}. \tag{6.9}$$

Then $\mathcal{O}$ is a closed, convex subset of $W^{1,1}(\Omega,\delta^\alpha)$, and by (6.8), $\mathbb{S}(\mathcal{O}) \subset \mathcal{O}$. Thus we can apply the Schauder fixed point theorem to obtain the existence of a function $v \in \mathcal{O}$ such that $\mathbb{S}(v) = v$. This means that v is a nonnegative solution of (6.1), and hence it holds

$$-\int_\Omega vL_\mu\zeta\,dx = \int_\Omega g(v+\varrho\mathbb{K}_\mu[\nu], |\nabla(v+\varrho\mathbb{K}_\mu[\nu])|)\zeta\,dx \quad \text{for every } \zeta \in \mathbf{X}_\mu(\Omega). \qquad \square$$

Proof of Theorem 1.9. Let $\{g_n\}$ be a sequence of C^1 nonnegative functions defined on $\mathbb{R}^2_+$ such that

$$g_n(0,0) = g(0,0) = 0, \quad g_n \le g_{n+1} \le g, \quad \sup_{\mathbb{R}\times\mathbb{R}_+} g_n = n \quad \text{and} \quad \lim_{n\to\infty} \|g_n - g\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}\times\mathbb{R}_+)=0}.$$

We observe that $\Lambda_{g_n} \le \Lambda_g < \infty$, where Λ_{g_n} is defined as in (1.10) with g replaced by g_n. Therefore, the constant ϱ_0 in Lemma 6.1 can be chosen to depend on Λ_g (and $N, \mu, \Omega, \tilde{k}, \tilde{p}, \tilde{q}$), but independent of n. Similarly, the constant t_0 in Lemma 6.1 can be chosen to depend on Λ_g (and also $N, \mu, \Omega, \tilde{k}, \tilde{p}, \tilde{q}$), but independent of n. By Lemma 6.1, for any $\varrho \in (0, \varrho_0)$ and $n \in \mathbb{N}$, there exists a solution $v_n \in \mathcal{O}$ (where $\mathcal{O}$ is defined in (6.9)) of

$$\begin{cases} -L_\mu v_n = g_n(v_n+\varrho\mathbb{K}_\mu[\nu], |\nabla(v_n+\varrho\mathbb{K}_\mu[\nu])|) & \text{in } \Omega, \\ \operatorname{tr}(v_n) = 0. \end{cases}$$

Set $u_n = v_n + \varrho\mathbb{K}_\mu[\nu]$. Then $\operatorname{tr}(u_n) = \varrho\nu$ and

$$-\int_\Omega u_nL_\mu\zeta\,dx = \int_\Omega g_n(u_n, |\nabla u_n|)\zeta\,dx - \varrho\int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta\,dx \quad \text{for every } \zeta \in \mathbf{X}_\mu(\Omega). \tag{6.10}$$

Since $\{v_n\} \subset \mathcal{O}$, the sequence $\{g_n(v_n+\varrho\mathbb{K}_\mu[\nu], |\nabla(v_n+\varrho\mathbb{K}_\mu[\nu])|)\}$ is uniformly bounded in $L^1(\Omega,\delta^\alpha)$, and the sequence $\{\frac{\mu}{\delta^2}v_n\}$ is uniformly bounded in $L^{p_1}(G)$ for every compact subset $G \subset \Omega$ for some $p_1 > 0$. As a consequence, $\{\Delta v_n\}$ is uniformly bounded in $L^1(G)$. By a standard regularity result for elliptic equations, $\{v_n\}$ is uniformly bounded in $W^{1,p_2}(G)$ for some $p_2 > 1$. Consequently, there exists a subsequence, still denoted by $\{v_n\}$, and a function v such that $v_n \to v$ a.e. in Ω and $\nabla v_n \to \nabla v$ a.e. in Ω. Therefore, $u_n \to u$ a.e. in Ω, where $u = v + \varrho\mathbb{K}_\mu[\nu]$ and $g_n(u_n, |\nabla u_n|) \to g(u, |\nabla u|)$ a.e. in Ω.

We show that $u_n \to u$ in $L^1(\Omega,\delta^\alpha)$. Since $\{v_n\}$ is uniformly bounded in $L^p(\Omega,\delta^\alpha)$, by (2.6), we derive that $\{u_n\}$ is uniformly bounded in $L^p(\Omega,\delta^\alpha)$. Due to Hölder's inequality, $\{u_n\}$ is equi-integrable in $L^1(\Omega,\delta^\alpha)$. We invoke Vitali's convergence theorem to derive that $u_n \to u$ in $L^1(\Omega,\delta^\alpha)$.

Next proceeding as in the proof of (3.11), we obtain that $g_n(u_n, |\nabla u_n|) \to g(u, |\nabla u|)$ in $L^1(\Omega,\delta^\alpha)$. Therefore, by sending $n \to \infty$ in each term of (6.10), we obtain

$$-\int_\Omega uL_\mu\zeta\,dx = \int_\Omega g(u, |\nabla u|)\zeta\,dx - \varrho\int_\Omega \mathbb{K}_\mu[\nu]L_\mu\zeta\,dx \quad \text{for every } \zeta \in \mathbf{X}_\mu(\Omega).$$

This means u is a nonnegative weak solution of ($P^{\varrho\nu}_-$). Therefore,

$$u = \mathbb{G}_\mu[g(u, |\nabla u|)] + \varrho\mathbb{K}_\mu[\nu] \quad \text{in } \Omega,$$

which implies that $u \ge \varrho\mathbb{K}_\mu[\nu]$ in Ω. $\square$

7 Nonlinear Equations with Supercritical Source

7.1 Capacities and Existence Results

In this subsection, we introduce the definition of some capacities and provide related results which will be employed to prove Theorem 1.11 in the next subsection.

For $0 \le \theta \le \beta < N$, set

$$N_{\theta,\beta}(x,y) := \frac{1}{|x-y|^{N-\beta}\max\{|x-y|,\,\delta(x),\,\delta(y)\}^{\theta}} \quad \text{for all } (x,y) \in \overline{\Omega}\times\overline{\Omega},\ x \neq y, \tag{7.1}$$

$$\mathbb{N}_{\theta,\beta}[\tau](x) := \int_{\overline{\Omega}} N_{\theta,\beta}(x,y)\,d\tau(y) \quad \text{for all } \tau \in \mathfrak{M}^+(\overline{\Omega}). \tag{7.2}$$

For $a > -1$, $0 \le \theta \le \beta < N$ and $s > 1$, define $\mathrm{Cap}^{a}_{\mathbb{N}_{\theta,\beta},s}$ by

$$\mathrm{Cap}^{a}_{\mathbb{N}_{\theta,\beta},s}(E) := \inf\Big\{\int_{\overline{\Omega}} \delta^{a}\phi^{s}\,dx : \phi \ge 0,\ \mathbb{N}_{\theta,\beta}[\delta^{a}\phi] \ge \chi_E\Big\} \quad \text{for any Borel set } E \subset \overline{\Omega}.$$

Here χ_E denotes the indicator function of E.

Let $\mathbf{Z}$ be a metric space and $\omega \in \mathfrak{M}^+(\mathbf{Z})$. Let $J : \mathbf{Z}\times\mathbf{Z} \to (0,\infty]$ be a Borel positive kernel such that J is symmetric and J^{-1} satisfies a quasi-metric inequality, i.e. there is a constant $C > 1$ such that, for all $x, y, z, \in \mathbf{Z}$,

$$\frac{1}{J(x,y)} \le C\Big(\frac{1}{J(x,z)} + \frac{1}{J(z,y)}\Big).$$

Under these conditions, one can define the quasi-metric d by

$$d(x,y) := \frac{1}{J(x,y)}$$

and denote by $\mathcal{B}_r(x) := \{y \in \mathbf{Z} : d(x,y) < r\}$ the open d-ball of radius $r > 0$ and center x. Note that this set can be empty.

For $\omega \in \mathfrak{M}^+(\mathbf{Z})$, we define the potentials $\mathbb{J}[\omega]$ and $\mathbb{J}[\phi,\omega]$ by

$$\mathbb{J}[\omega](x) := \int_{\mathbf{Z}} J(x,y)\,d\omega(y) \quad \text{and} \quad \mathbb{J}[\phi,\omega](x) := \int_{\mathbf{Z}} J(x,y)\phi(y)\,d\omega(y).$$

For $t > 1$, the capacity $\mathrm{Cap}^{\omega}_{\mathbb{J},t}$ in $\mathbf{Z}$ is defined by

$$\mathrm{Cap}^{\omega}_{\mathbb{J},t}(E) := \inf\Big\{\int_{\mathbf{Z}} \phi(x)^{t}\,d\omega(x) : \phi \ge 0,\ \mathbb{J}[\phi,\omega] \ge \chi_E\Big\} \quad \text{for any Borel } E \subset \mathbf{Z}.$$

Proposition 7.1 ([19]). *Let $p > 1$ and $\tau, \omega \in \mathfrak{M}^+(\mathbf{Z})$ such that*

$$\int_0^{2r} \frac{\omega(\mathcal{B}_s(x))}{s^2}\,ds \le C\int_0^{r} \frac{\omega(\mathcal{B}_s(x))}{s^2}\,ds, \tag{7.3}$$

$$\sup_{y\in\mathcal{B}_r(x)} \int_0^{r} \frac{\omega(\mathcal{B}_s(y))}{s^2}\,ds \le C\int_0^{r} \frac{\omega(\mathcal{B}_s(x))}{s^2}\,ds \tag{7.4}$$

for any $r > 0$, $x \in \mathbf{Z}$, where $C > 0$ is a constant. Then the following statements are equivalent.

(1) *The equation $u = \mathbb{J}[u^p,\omega] + \sigma\mathbb{J}[\tau]$ has a solution for $\sigma > 0$ small.*
(2) *For any Borel set $E \subset \mathbf{Z}$, it holds $\int_E \mathbb{J}[\tau_E]^p\,d\omega \le C\tau(E)$, where $\tau_E = \chi_E\tau$.*
(3) *For any Borel set $E \subset \mathbf{Z}$, it holds $\tau(E) \le C\,\mathrm{Cap}^{\omega}_{\mathbb{J},p'}(E)$.*
(4) *The inequality $\mathbb{J}[\mathbb{J}[\tau]^p,\omega] \le C\mathbb{J}[\tau] < \infty$ holds ω-a.e.*

We point out below that $\mathbb{N}_{\theta,\beta}$ defined in (7.2) satisfies all assumptions of $\mathbb{J}$ in Proposition 7.1.

Proposition 7.2 ([9, Lemma 2.2]). *$N_{\theta,\beta}$ is symmetric and satisfies the quasi-metric inequality.*

Next we give sufficient conditions for (7.3), (7.4) to hold.

Proposition 7.3. *Let $\omega = \delta(x)^a\chi_\Omega(x)\,dx$ with $a > -1$. Then* (7.3) *and* (7.4) *hold.*

Proof. If $a \ge 0$, then the statement follows from [9, Lemma 2.3]. We now treat the case $-1 < a < 0$. We claim that, for any $0 < s < 8\,\mathrm{diam}(\overline{\Omega})$ and any $x \in \overline{\Omega}$, we have

$$\omega(B_s(x)) \approx \max\{\delta(x), s\}^a s^N. \tag{7.5}$$

Indeed, in order to obtain (7.5), we consider four cases.

Case 1: $4s \le \delta(x)$. Then $\delta(x) \approx \delta(y)$ for any $y \in B_s(x)$, and the proof of (7.5) can be obtained easily.

Case 2: $s > \frac{\delta(x)}{4}$. Then $\delta(y) \le 5s$; thus

$$\int_{B_s(x)\cap\overline{\Omega}} \delta(y)^a \, dy \ge Cs^{a+N} \ge C \max\{\delta(x), s\}^a s^N.$$

Case 3: $\frac{\delta(x)}{4} \le s \le 4\delta(x)$. Since Ω is smooth, there exists $r^* > 0$ such that

$$\int_{B_{r_0}(x_i)\cap\overline{\Omega}} \delta(y)^a \, dy \le \frac{C}{a+1} r_0^{a+N} \quad \text{for all } r_0 \le \frac{r^*}{8} \text{ and } \delta(x_i) < \frac{r^*}{4}. \tag{7.6}$$

Set

$$r_0 := r^* \frac{\delta(x)}{32 \operatorname{diam}(\overline{\Omega})}.$$

Then there exist $x_i \in B_s(x)$, $i = 1, \dots, k$, such that $B_s(x) \subset \bigcup_{i=1}^k B_{r_0}(x_i)$. We note that k does depend neither on x, nor on $\delta(x)$. Thus we have

$$\int_{B_s(x)\cap\overline{\Omega}} \delta(y)^a \, dy \le \sum_{i=1}^k \int_{B_{r_0}(x_i)\cap\overline{\Omega}} \delta(y)^a \, dy.$$

Now, by (7.6), we get

$$\int_{B_{r_0}(x_i)\cap\overline{\Omega}} \delta(y)^a \, dy \le C\delta(x)^{a+N} \le C \max\{\delta(x), s\}^a s^N \qquad \text{if } \delta(x_i) < \frac{r^*}{4},$$

$$\int_{B_{r_0}(x_i)\cap\overline{\Omega}} \delta(y)^a \, dy \le C(r^*)^a \delta(x)^N \le C \max\{\delta(x), s\}^a s^N \quad \text{if } \delta(x_i) \ge \frac{r^*}{4},$$

and hence (7.5) follows.

Case 4: $s \ge 4\delta(x)$. Set

$$r_0 := r^* \frac{s}{32 \operatorname{diam}(\overline{\Omega})}.$$

Then the proof of (7.5) follows due to an argument similar to Case 3.

The rest of the proof can proceed as in the proof of [9, Lemma 2.3], and we omit it. □

We recall below the definition of the capacity associated to $\mathbb{N}_{\theta,\beta}$ (see [19]).

Definition 7.4. Let $a > -1$, $0 \le \theta \le \beta < N$ and $s > 1$. For any Borel set $E \subset \overline{\Omega}$, define $\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}$ by

$$\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E) := \inf\Big\{ \int_{\overline{\Omega}} \delta^a \phi^s \, dy : \phi \ge 0,\ \mathbb{N}_{\theta,\beta}[\delta^a \phi] \ge \chi_E \Big\}.$$

Clearly, for any Borel set $E \subset \overline{\Omega}$, we have

$$\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E) = \inf\Big\{ \int_{\overline{\Omega}} \delta^{-a(s-1)} \phi^s \, dy : \phi \ge 0,\ \mathbb{N}_{\theta,\beta}[\phi] \ge \chi_E \Big\}.$$

Furthermore, by [1, Theorem 2.5.1], we have

$$(\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E))^{\frac{1}{s}} = \inf\{\omega(E) : \omega \in \mathfrak{M}^+_b(\overline{\Omega}),\ \|\mathbb{N}_{\theta,\beta}[\omega]\|_{L^{s'}(\overline{\Omega};\delta^a)} \le 1\}$$

for any compact set $E \subset \overline{\Omega}$, where s' is the conjugate exponent of s.

Thanks to Propositions 7.2 and 7.3, we can apply Proposition 7.1 to obtain the following result.

Proposition 7.5. *Let $\tau \in \mathfrak{M}^+(\overline{\Omega})$, $a > -1$, $0 \le \theta \le \beta < N$, $p > 1$. Then the following statements are equivalent.*

(1) *For any Borel set $E \subset \overline{\Omega}$, it holds $\tau(E) \le C\,\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},p'}(E)$.*

(2) *The inequality $\mathbb{N}_{\theta,\beta}[\delta^a \mathbb{N}_{\theta,\beta}[\tau]^p] \le C\mathbb{N}_{\theta,\beta}[\tau] < \infty$ holds a.e. in Ω.*

Recall the capacity $\mathrm{Cap}^{\partial\Omega}_{\theta,s}$ introduced in [9] which is used to deal with boundary measures. Let $\theta \in (0, N-1)$, and denote by $\mathcal{B}_\theta$ the Bessel kernel in $\mathbb{R}^{N-1}$ with order θ. For $s > 1$, define

$$\mathrm{Cap}_{\mathcal{B}_\theta,s}(F) := \inf\Big\{ \int_{\mathbb{R}^{N-1}} \phi^s\, dy : \phi \geq 0,\ \mathcal{B}_\theta * \phi \geq \chi_F \Big\} \quad \text{for any Borel set } F \subset \mathbb{R}^{N-1}.$$

Since Ω is a bounded smooth domain in $\mathbb{R}^N$, there exist open sets $O_1, \dots, O_m$ in $\mathbb{R}^N$, diffeomorphisms $T_i\colon O_i \to B_1(0)$ and compact sets $K_1, \dots, K_m$ in $\partial\Omega$ such that

(i) $K_i \subset O_i$, $1 \leq i \leq m$, and $\partial\Omega \subset \bigcup_{i=1}^m K_i$,

(ii) $T_i(O_i \cap \partial\Omega) = B_1(0) \cap \{x_N = 0\}$, $T_i(O_i \cap \Omega) = B_1(0) \cap \{x_N > 0\}$,

(iii) for any $x \in O_i \cap \Omega$, there exists $y \in O_i \cap \partial\Omega$ such that $\delta(x) = |x - y|$.

We then define the $\mathrm{Cap}^{\partial\Omega}_{\theta,s}$-capacity of a compact set $F \subset \partial\Omega$ by

$$\mathrm{Cap}^{\partial\Omega}_{\theta,s}(F) := \sum_{i=1}^m \mathrm{Cap}_{\mathcal{B}_\theta,s}(\tilde{T}_i(F \cap K_i)),$$

where $T_i(F \cap K_i) = \tilde{T}_i(F \cap K_i) \times \{x_N = 0\}$.

The following result is obtained by the same argument as in the proof of [9, Proposition 2.9].

Proposition 7.6. *Let $a > -1$, $0 \leq \theta \leq \beta < N$ and $s > 1$. Assume that $-1 + s'(1 + \theta - \beta) < a < -1 + s'(N + \theta - \beta)$. Then it holds*

$$\mathrm{Cap}^a_{\mathbb{N}_{\theta,\beta},s}(E) \approx \mathrm{Cap}^{\partial\Omega}_{\beta-\theta+\frac{a+1}{s'}-1,s}(E) \quad \textit{for any Borel } E \subset \partial\Omega.$$

7.2 Case $g(u, |\nabla u|) = |u|^p |\nabla u|^q$

Proof of Theorem 1.11. We see that, under the assumption on p and q, from Proposition 7.5 and Proposition 7.6, conditions (i) and (ii) are equivalent. Therefore, we will prove the existence of a solution by assuming (ii). For $u \in W^{1,1}_{\mathrm{loc}}(\Omega)$, put

$$\mathbb{H}[u](x) := \mathbb{G}_\mu[|u|^p |\nabla u|^q](x) + \mathbb{K}_\mu[\varrho\nu](x) \quad \text{a.e. in } \Omega.$$

From (2.1), (2.2) and (7.1), we have

$$\begin{aligned} G_\mu(x,y) &\leq C_1 \delta(x)^\alpha \delta(y)^\alpha N_{2\alpha,2}(x,y) \leq C_1 \delta(x)^\alpha \delta(y)^\alpha N_{2\alpha-1,1}(x,y) && \text{for all } x, y \in \Omega,\ x \neq y,\\ K_\mu(x,y) &\leq C_1 \delta(x)^\alpha N_{2\alpha-1,1}(x,y) && \text{for all } x \in \Omega,\ y \in \partial\Omega, \end{aligned} \tag{7.7}$$

and

$$\begin{aligned} |\nabla_x G_\mu(x,y)| &\leq C_1 \delta(x)^{\alpha-1} \delta(y)^\alpha N_{2\alpha-1,1}(x,y) && \text{for all } x, y \in \Omega,\ x \neq y,\\ |\nabla_x K_\mu(x,y)| &\leq C_1 \delta(x)^{\alpha-1} N_{2\alpha-1,1}(x,y) && \text{for all } x \in \Omega,\ y \in \partial\Omega. \end{aligned} \tag{7.8}$$

From (7.7) and (7.8), we obtain

$$\begin{aligned} |\mathbb{H}[u]| &\leq C_1 \delta^\alpha \mathbb{N}_{2\alpha-1,1}[\delta^\alpha |u|^p |\nabla u|^q] + C_1 \delta^\alpha \mathbb{N}_{2\alpha-1,1}[\varrho\nu],\\ |\nabla \mathbb{H}[u]| &\leq C_1 \delta^{\alpha-1} \mathbb{N}_{2\alpha-1,1}[\delta^\alpha |u|^p |\nabla u|^q] + C_1 \delta^{\alpha-1} \mathbb{N}_{2\alpha-1,1}[\varrho\nu]. \end{aligned}$$

Put

$$\mathcal{E} := \{u \in W^{1,1}_{\mathrm{loc}}(\Omega) : |u| \leq 2C_1 \delta^\alpha \mathbb{N}_{2\alpha-1,1}[\varrho\nu],\ |\nabla u| \leq 2C_1 \delta^{\alpha-1} \mathbb{N}_{2\alpha-1,1}[\varrho\nu]\}.$$

Then, by using (1.22), we deduce that there exists $\varrho_0 = \varrho_0(p, q, C_1, C) > 0$ such that if $\varrho \in (0, \varrho_0)$, then $\mathbb{H}(\mathcal{E}) \subset \mathcal{E}$.

Define $\mathcal{V}$ the space of functions $v \in W^{1,1}_{\mathrm{loc}}(\Omega)$ with the norm

$$\|v\|_{\mathcal{V}} = \|v\|_{L^{p+q}(\Omega,\delta^{-q+\alpha})} + \|\nabla v\|_{L^{p+q}(\Omega,\delta^{p+\alpha})}.$$

We can see that $\mathcal{E} \subset \mathcal{V}$ and $\mathcal{E}$ is convex and closed under the strong topology of $\mathcal{V}$. Moreover, it can be justified that $\mathbb{H}$ is a continuous and compact operator. Therefore, by invoking the Schauder fixed point theorem, we conclude that there exists $u \in \mathcal{E}$ such that $\mathbb{H}[u] = u$. Therefore, u is a weak solution of problem $(P^{\varrho\nu}_-)$ satisfying (1.23) with $C' = 2C_1$. $\square$

A Barrier

In this section, we will provide a barrier which plays an important role. This barrier will have the same properties as the barrier in [17, Proposition 6.1]. Let β_0 be the constant in Proposition 2.6.

Proposition A.1. *Let $\Omega \subset \mathbb{R}^N$ be a C^2 domain, $0 < \mu \leq \frac{1}{4}$, $q > 0$ and $p + q > 1$. Then, for any $z \in \partial\Omega$ and $0 < R \leq \frac{\beta_0}{16}$, there exists a supersolution $w := w_{z,R}$ of (4.2) in $\Omega \cap B_R(z)$ such that $w \in C(\overline{\Omega} \cap B_R(z))$, $w(x) \to \infty$ when $\operatorname{dist}(x, K) \to 0$, for any compact subset $K \subset \Omega \cap \partial B_R(z)$, and w vanishes on $\partial\Omega \cap B_R(z)$. More precisely,*

$$w(x) = \begin{cases} c(R^2 - |x-z|^2)^{-b}\delta(x)^{\gamma} & \text{for all } \gamma \in (1-\alpha, \alpha) \quad \text{if } 0 < \mu < \frac{1}{4}, \\ c(R^2 - |x-z|^2)^{-b}\delta(x)^{\frac{1}{2}}\big(\ln \frac{\operatorname{diam}(\Omega)}{\delta(x)}\big)^{\frac{1}{2}} & \text{if } \mu = \frac{1}{4}, \end{cases}$$

where b is a constant such that $b \geq \max\{\frac{4-q-p}{q+p-1} + \gamma, \frac{N-2}{2}, 1\}$ and $c = c(N, \mu, p, q, b, \gamma)$.

Proof. The proof is similar to that of [17, Proposition 6.1] with some minor modifications, and hence we omit it. □

B Case $g(u, |\nabla u|) = |u|^p + |\nabla u|^q$

In this section, we assume that $g(u, |\nabla u|) = |u|^p + |\nabla u|^q$ with $p > 1$ and $1 < q < 2$. We will state main results for this case without proving since the proofs are similar, even simpler, to those for the case $g(u, |\nabla u|) = |u|^p|\nabla u|^q$.

B.1 Absorption Case

This subsection is devoted to the study of the equation

$$-L_\mu u + |u|^p + |\nabla u|^q = 0 \quad \text{in } \Omega. \tag{B.1}$$

When $g(u, |\nabla u|) = |u|^p + |\nabla u|^q$ with $p, q > 1$, then g satisfies (1.10) if p and q satisfy (1.12). Moreover, g satisfies (1.21). Hence, if p and q satisfy (1.12), then, for any $\nu \in \mathfrak{M}^+(\partial\Omega)$, the problem

$$\begin{cases} -L_\mu u + |u|^p + |\nabla u|^q = 0 & \text{in } \Omega, \\ \operatorname{tr}(\nu) = \nu \end{cases} \tag{B.2}$$

admits a positive weak solution.

Theorem B.1. *Assume p and q satisfy (1.12). Let $\nu_i \in \mathfrak{M}^+(\partial\Omega)$, $i = 1, 2$, and let u_i be a nonnegative solution of (B.2) with $\nu = \nu_i$. If $\nu_1 \leq \nu_2$, then $u_1 \leq u_2$ in Ω.*

Set

$$m_{p,q} := \max\Big\{p, \frac{q}{2-q}\Big\}.$$

Lemma B.2. *Let $p > 1$ and $1 < q < \frac{N}{N-1}$. If u is a nonnegative solution of (B.1), then*

$$u(x) \leq C\delta(x)^{-\frac{2}{m_{p,q}-1}} \quad \text{for all } x \in \Omega,$$
$$|\nabla u(x)| \leq C\delta(x)^{-\frac{2}{m_{p,q}-1}-1} \quad \text{for all } x \in \Omega.$$

Lemma B.3. *Let p and q satisfy (1.12). Assume u is a positive solution of (B.1) in Ω such that (4.14) holds locally uniformly in $\partial\Omega \setminus \{0\}$. Then there exists a constant $C = C(N, \mu, p, q, \Omega)$ such that*

$$u(x) \leq C\delta(x)^{\alpha}|x|^{-\frac{2}{m_{p,q}-1}-\alpha} \quad \text{for all } x \in \Omega,$$
$$|\nabla u(x)| \leq C\delta(x)^{\alpha-1}|x|^{-\frac{2}{m_{p,q}-1}-\alpha} \quad \text{for all } x \in \Omega.$$

Theorem B.4. *Assume $g(u, |\nabla u|) = |u|^p + |\nabla u|^q$ with p and q satisfying* (1.12).

(I) Weak singularity. *For any $k > 0$, let $u_{0,k}^{\Omega}$ be the solution of* (1.13). *Then* (1.14) *holds. Furthermore, the mapping $k \mapsto u_{0,k}^{\Omega}$ is increasing.*

(II) Strong singularity. *Put $u_{0,\infty}^{\Omega} := \lim_{k\to\infty} u_{0,k}^{\Omega}$. Then $u_{0,\infty}^{\Omega}$ is a solution of* (1.15). *Then there exists a constant $c = c(N, \mu, p, q, \Omega) > 0$ such that*

$$c^{-1}\delta(x)^{\alpha}|x|^{-\frac{2}{m_{p,q}-1}} \le u_{0,\infty}^{\Omega}(x) \le c\delta(x)^{\alpha}|x|^{-\frac{2}{m_{p,q}-1}} \qquad \text{for all } x \in \Omega,$$
$$|\nabla u_{0,\infty}^{\Omega}(x)| \le c\delta(x)^{\alpha-1}|x|^{-\frac{2}{m_{p,q}-1}-\alpha} \qquad \text{for all } x \in \Omega.$$

Moreover,

$$\lim_{\substack{\Omega\ni x\to 0\\ \frac{x}{|x|}=\sigma\in S_+^{N-1}}} |x|^{\frac{2}{m_{p,q}-1}} u_{0,\infty}^{\Omega}(x) = \tilde{\omega}(\sigma)$$

locally uniformly on upper hemisphere $S_+^{N-1} = \mathbb{R}_+^N \cap S^N$. Here $\tilde{\omega}$ is the unique positive solution of

$$\begin{cases} -\mathcal{L}_{\mu}\omega - \ell_{N,p,q}\omega + J(\omega, \nabla'\omega) = 0 & \text{in } S_+^{N-1}, \\ \omega = 0 & \text{on } \partial S_+^{N-1}, \end{cases}$$

where

$$\mathcal{L}_{\mu}\omega := \Delta'\omega + \frac{\mu}{(\mathbf{e}_N\cdot\sigma)^2}\omega, \quad \ell_{N,p,q} := \frac{2}{m_{p,q}}\left(\frac{2}{m_{p,q}} + 2 - N\right),$$

$$J(s,\xi) := \begin{cases} \left(\left(\frac{2}{m_{p,q}}\right)^2 s^2 + |\xi|^2\right)^{\frac{q}{2}} & \text{if } p < \frac{q}{2-q},\ (s,\xi) \in \mathbb{R}_+\times\mathbb{R}^N, \\ s^p + \left(\left(\frac{2}{m_{p,q}}\right)^2 s^2 + |\xi|^2\right)^{\frac{q}{2}} & \text{if } p = \frac{q}{2-q},\ (s,\xi) \in \mathbb{R}_+\times\mathbb{R}^N, \\ s^p & \text{if } p > \frac{q}{2-q},\ (s,\xi) \in \mathbb{R}_+\times\mathbb{R}^N. \end{cases}$$

Theorem B.5. *Let $\nu \in \mathfrak{M}^+(\partial\Omega)$, $p \ge p_\mu$ or $q_\mu \le q < 2$. Assume problem* (B.2) *admits a weak solution.*

(I) *If $p \ge p_\mu$, then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{2-\frac{1+\alpha}{p'},p'}$.*

(II) *If $q_\mu \le q < 2$, then the following occurs.*

   (i) *If $q \ne \alpha + 1$, then ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{\frac{\alpha+1}{q}-\alpha,q'}$.*

   (ii) *If $q = \alpha + 1$, then, for any $\varepsilon \in (0, \min\{\alpha+1, \frac{(N-1)\alpha}{\alpha+1} - (1-\alpha)\})$, ν is absolutely continuous with respect to $C^{\mathbb{R}^{N-1}}_{\varepsilon+1-\alpha,\frac{\alpha+1}{\alpha}}$.*

Theorem B.6. *Assume $p \ge p_\mu$ or $q_\mu \le q < 2$. Let $K \subset \partial\Omega$ be compact such that one of the following holds:*

$$C^{\mathbb{R}^{N-1}}_{2-\frac{1+\alpha}{p'},p'}(K) = 0 \quad \text{if } p \ge p_\mu,$$
$$C^{\mathbb{R}^{N-1}}_{\frac{\alpha+1}{q}-\alpha,q'}(K) = 0 \quad \text{if } q_\mu \le q < 2 \text{ and } q \ne \alpha+1,$$
$$C^{\mathbb{R}^{N-1}}_{\varepsilon+1-\alpha,q'}(K) = 0 \quad \text{if } q = \alpha+1 \quad \text{for some } \varepsilon \in \left(0, \min\left\{\alpha+1, \frac{(N-1)\alpha}{\alpha+1} - (1-\alpha)\right\}\right).$$

Then any nonnegative solution $u \in C^2(\Omega) \cap C(\overline{\Omega}\setminus K)$ of equation (B.1) *satisfying* (1.20) *is identically zero.*

B.2 Source Case

Theorem B.7. *Assume $g(u, |\nabla u|) = |u|^p + |\nabla u|^q$ with $p > 1$ and $\frac{\alpha+1}{N+\alpha-1} < q < \frac{1+\alpha}{\alpha}$. Assume one of the following conditions holds.*

(i) *There exists a constant $C > 0$ such that, for every Borel set $E \subset \partial\Omega$,*

$$\nu(E) \le C\min\{\mathrm{Cap}^{\partial\Omega}_{1-\alpha+\frac{\alpha+1}{p},p'}(E), \mathrm{Cap}^{\partial\Omega}_{-\alpha+\frac{\alpha+1}{q},q'}(E)\}.$$

(ii) *There exists a positive constant $C > 0$ such that*

$$\mathbb{N}_{2\alpha,2}[\delta^{\alpha(p+1)}\mathbb{N}_{2\alpha,2}[\nu]^p] \le C\mathbb{N}_{2\alpha,2}[\nu] < \infty \qquad \text{a.e. in } \Omega,$$
$$\mathbb{N}_{2\alpha-1,1}[\delta^{(\alpha-1)q+\alpha}\mathbb{N}_{2\alpha-1,1}[\nu]^q] \le C\mathbb{N}_{2\alpha-1,1}[\nu] < \infty \qquad \text{a.e. in } \Omega.$$

Then there exists $\varrho_0 = \varrho_0(N, \mu, p, q, C, \Omega) > 0$ *such that if* $\varrho \in (0, \varrho_0)$, *then problem* $(P^{\varrho\nu}_{-})$ *admits a weak solution u satisfying* (1.23).

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References

[1] D. R. Adams and L. I. Hedberg, *Function Spaces and Potential Theory*, Grundlehren Math. Wiss. 314, Springer, Berlin, 1996.
[2] A. Aghajani, C. Cowan and S. H. Lui, Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball, *J. Differential Equations* **264** (2018), no. 4, 2865–2896.
[3] A. Ancona, Negatively curved manifolds, elliptic operators, and the Martin boundary, *Ann. of Math. (2)* **125** (1987), no. 3, 495–536.
[4] C. Bandle, M. Marcus and V. Moroz, Boundary singularities of solutions of semilinear elliptic equations in the half-space with a Hardy potential, *Israel J. Math.* **222** (2017), no. 1, 487–514.
[5] C. Bandle, V. Moroz and W. Reichel, "Boundary blowup" type sub-solutions to semilinear elliptic equations with Hardy potential, *J. Lond. Math. Soc. (2)* **77** (2008), no. 2, 503–523.
[6] G. Barbatis, S. Filippas and A. Tertikas, A unified approach to improved L^p Hardy inequalities with best constants, *Trans. Amer. Math. Soc.* **356** (2004), no. 6, 2169–2196.
[7] M. F. Bidaut-Véron, M. Garcia-Huidobro and L. Véron, A priori estimates for elliptic equations with reaction terms involving the function and its gradient, preprint (2018), https://arxiv.org/abs/1810.12230.
[8] M.-F. Bidaut-Véron, M. García-Huidobro and L. Véron, Estimates of solutions of elliptic equations with a source reaction term involving the product of the function and its gradient, *Duke Math. J.* **168** (2019), no. 8, 1487–1537.
[9] M.-F. Bidaut-Véron, G. Hoang, Q.-H. Nguyen and L. Véron, An elliptic semilinear equation with source term and boundary measure data: The supercritical case, *J. Funct. Anal.* **269** (2015), no. 7, 1995–2017.
[10] H. Brezis, Une équation semilinéaire avec conditions aux limites dans L^1, unpublished note (1972).
[11] H. Brezis and M. Marcus, Hardy's inequalities revisited, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (4)* no. 25 (1997), 217–237.
[12] L. D'Ambrosio and S. Dipierro, Hardy inequalities on Riemannian manifolds and applications, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **31** (2014), no. 3, 449–475.
[13] J. Dávila and L. Dupaigne, Hardy-type inequalities, *J. Eur. Math. Soc. (JEMS)* **6** (2004), no. 3, 335–365.
[14] S. Filippas, L. Moschini and A. Tertikas, Sharp two-sided heat kernel estimates for critical Schrödinger operators on bounded domains, *Comm. Math. Phys.* **273** (2007), no. 1, 237–281.
[15] K. T. Gkikas and P.-T. Nguyen, On the existence of weak solutions of semilinear elliptic equations and systems with Hardy potentials, *J. Differential Equations* **266** (2019), no. 1, 833–875.
[16] K. T. Gkikas and P. T. Nguyen, Semilinear elliptic equations with Hardy potential and gradient nonlinearity, preprint (2019), https://arxiv.org/abs/1903.11090; to appear in Rev. Mat. Iberoam.
[17] K. T. Gkikas and L. Véron, Boundary singularities of solutions of semilinear elliptic equations with critical Hardy potentials, *Nonlinear Anal.* **121** (2015), 469–540.
[18] A. Gmira and L. Véron, Boundary singularities of solutions of some nonlinear elliptic equations, *Duke Math. J.* **64** (1991), no. 2, 271–324.
[19] N. J. Kalton and I. E. Verbitsky, Nonlinear equations and weighted norm inequalities, *Trans. Amer. Math. Soc.* **351** (1999), no. 9, 3441–3497.
[20] J. L. Kazdan and R. J. Kramer, Invariant criteria for existence of solutions to second-order quasilinear elliptic equations, *Comm. Pure Appl. Math.* **31** (1978), no. 5, 619–645.
[21] M. Marcus, Estimates of Green and Martin kernels for Schrödinger operators with singular potential in Lipschitz domains, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **36** (2019), no. 5, 1183–1200.
[22] M. Marcus, V. J. Mizel and Y. Pinchover, On the best constant for Hardy's inequality in $\mathbf{R}^n$, *Trans. Amer. Math. Soc.* **350** (1998), no. 8, 3237–3255.
[23] M. Marcus and V. Moroz, Moderate solutions of semilinear elliptic equations with Hardy potential under minimal restrictions on the potential, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **18** (2018), no. 1, 39–64.
[24] M. Marcus and P.-T. Nguyen, Elliptic equations with nonlinear absorption depending on the solution and its gradient, *Proc. Lond. Math. Soc. (3)* **111** (2015), no. 1, 205–239.
[25] M. Marcus and P.-T. Nguyen, Moderate solutions of semilinear elliptic equations with Hardy potential, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **34** (2017), no. 1, 69–88.

[26] M. Marcus and L. Véron, The boundary trace of positive solutions of semilinear elliptic equations: The subcritical case, *Arch. Ration. Mech. Anal.* **144** (1998), no. 3, 201–231.

[27] M. Marcus and L. Veron, The boundary trace of positive solutions of semilinear elliptic equations: The supercritical case, *J. Math. Pures Appl. (9)* **77** (1998), no. 5, 481–524.

[28] M. Marcus and L. Veron, Removable singularities and boundary traces, *J. Math. Pures Appl. (9)* **80** (2001), no. 9, 879–900.

[29] M. Marcus and L. Véron, *Nonlinear Second Order Elliptic Equations Involving Measures*, De Gruyter Ser. Nonlinear Anal. Appl. 21, De Gruyter, Berlin, 2014.

[30] P.-T. Nguyen, Isolated singularities of positive solutions of elliptic equations with weighted gradient term, *Anal. PDE* **9** (2016), no. 7, 1671–1692.

[31] P.-T. Nguyen, Semilinear elliptic equations with Hardy potential and subcritical source term, *Calc. Var. Partial Differential Equations* **56** (2017), no. 2, Article ID 44.

[32] T. Nguyen Phuoc and L. Véron, Boundary singularities of solutions to elliptic viscous Hamilton–Jacobi equations, *J. Funct. Anal.* **263** (2012), no. 6, 1487–1538.

[33] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton University, Princeton, 1970.

[34] L. Véron, *Local and Global Aspects of Quasilinear Degenerate Elliptic Equations. Quasilinear Elliptic Singular Problems*, World Scientific, Hackensack, 2017.